Introduction to Astrophysics
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Stability requires that $p(r)/\rho^{\Gamma}(r)$ increases with r. If that is not the case convection occurs. In an ideal gas $p \propto \rho T$ so we get $p(r)/\rho^{\Gamma}(r) \propto T^{\Gamma}/p^{(\Gamma-1)}$.
Then for adiabatic variations $T \propto p^{(\Gamma-1)/\Gamma}$. Then the quantity $\nabla_{ad}(r)$ in eq (217)

$$\nabla_{ad}(r) \equiv \left[\frac{\partial \ln T(p)}{\partial \ln p}\right]_{p=p(r)},$$

is the constant quantity

$$\nabla_{ad} = 1 - 1/\Gamma \tag{230}$$

which is the value that satisfies the requirement (219)

$$\nabla(r) < \nabla_{ad}(r).$$

For a monatomic ideal gas of atoms at temperature T the equipartition of energy gives a thermal energy per atom $3k_BT/2$. With $\rho/m_1\mu$ atoms per volume (where μ is the atomic weight and m_1 the mass for unit atomic weight), the thermal energy per volume is $E = 3k_BT\rho/2\mu m_1$. If we compared with a pressure given by the ideal gas law as $p = k_BT\rho/\mu m_1$, we can see that the polytropic equation $\mathcal{E} = p/(\Gamma - 1)$ is satisfied with $\Gamma = 1 + 2/3 = 5/3$, and $\nabla_{ad} = 2/5$.

The physics in actual stars, even in ordinary stars is not so simple.

When going inwards in the Sun from just below the surface to $r \simeq 0.8R_{\odot}$, the increasing temperature goes first to ionizing atomic hydrogen (which takes 13.6 eV per atom), then to singly ionizing atomic helium (24.6 eV per atom), and then to completely ionizing singly ionized helium (54.4 eV per ion), rather than to increasing thermal velocities and pressure. Since $\partial E/\partial p$ is thus effectively greater than 3/2, the effective value of Γ is less than 5/3, and ∇_{ad} is less than 2/5. In the outer layers of the Sun, from just below the surface down to $r \simeq 0.8R_{\odot}$, the effective value of ∇_{ad} is approximately 0.15. Elsewhere in the Sun, ∇_{ad} is close to the nominal value 2/5. Energy density is proportional to pressure also if the thermal energy and pressure are both dominated by relativistic particles, such as fast electrons in high-mass white dwarfs or photons in supermassive stars. In such cases we have $p = E/3$, so the equations are satisfied with $\Gamma = 4/3$ and $\nabla_{ad} = 1/4$. Now suppose that, in some part of a star, the condition (216) for convective stability is not satisfied, but instead we have

$$\left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r) < \rho'(r). \tag{231}$$

or

$$\nabla(r) > \nabla_{ad}(r). \tag{232}$$

This is the case in the Sun from just below the surface, at a depth where$p \cong 10^5 dyne/cm^2$, down to $r \cong 0.7R_\odot$, where $p \cong 10^{13.5} dyne/cm^2$.) In this case a blob of stellar fluid that happens to move upwards or downwards will become respectively lighter or heavier than the same volume of ambient fluid along its path, and hence will tend to keep moving in the same direction. The pressure in the blob remains the same as the ambient pressure along its path, so if the energy per volume E depends only on the pressure, it too remains the same in the blob as in the fluid along its path, but since the mass density ρ in the blob becomes less or greater than in the fluid along its path for a blob going upwards or downwards, the energy per mass $E(p)/\rho$ becomes respectively greater or less than in the fluid along its path. Specifically, after the blob travels a distance δr, the difference between its density and the density of the surrounding material will be

$$\delta\rho = \left[\left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r) - \rho'(r)\right]\delta r. \tag{233}$$

so the difference between the thermal energy per mass of the blob and of the surrounding material will be

$$\delta\left(\frac{\mathcal{E}}{\rho}\right) = -\mathcal{E}\delta\rho/\rho^2 = \frac{\mathcal{E}(r)}{\rho^2(r)}\left[\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right]\delta r. \quad (234)$$

According to the condition (231) for convection to occur, the change (234) in energy per mass of the blob will be positive or negative for outward or inward motion, respectively. Eventually the blob will dissolve into the ambient material, heating the ambient matter above if the blob has gone upward and cooling the matter below if the blob has gone downward. The succession of blobs going up and down thus leads to a flow of heat outward. We will refer to the rate of energy transport outward through a sphere of radius r by radiation and convection as $\mathcal{L}_{rad}(r)$ and $\mathcal{L}_{conv}(r)$, respectively, while the total rate of energy transport is

$$\mathcal{L}_{rad}(r) + \mathcal{L}_{conv}(r) \equiv \mathcal{L}_{tot}(r) \quad (235)$$

The equation (of energy conservation is

$$\frac{d\mathcal{L}_{tot}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r), \quad (236)$$

while it is $\mathcal{L}_{rad}(r)$ the one involved in temperature through Eq. (107), which we now write as

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)\mathcal{L}_{rad}(r)}{4acT^3(r)4\pi r^2} \quad (237)$$

When there is convection, Eq. (220) refers to the radiative energy transport rate, not the total rate:

$$\nabla(r) = \frac{3\kappa(r)\mathcal{L}_{rad}(r)p(r)}{16\pi caT^4(r)G\mathcal{M}(r)}. \quad (238)$$

More properly $\nabla_{rad}(r)$ is what $\nabla(r)$ would be if all the energy transferred was entirely transported by radiation.

$$\nabla_{rad}(r) \equiv \frac{3\kappa(r)\mathcal{L}_{tot}(r)p(r)}{16\pi caT^4(r)G\mathcal{M}(r)}. \tag{239}$$

Convection does carry some energy. Its presence means that $\nabla(r)$ is somehow less than $\nabla_{rad}(r)$ (typically much less), but also greater than the adiabatic component $\nabla_{ad}(r)$. When the energy transferred is originated in nuclear reactions (236) holds. But outside the central core where this energy generation takes place $\mathcal{L}_{tot}(r)$ is a constant and then equal to the star's luminosity L. To calculate the variation in the star's temperature, we need to find $\mathcal{L}_{rad}(r)$, which is not simple. Instead we can often simply "assume that in convective zones $\nabla(r) \simeq \nabla_{ad}(r)$, so that the temperature varies in such a way as to keep the pressure simply proportional to ρ^{Γ}. To see when this is likely to be the case, it is usual to calculate the convective energy flux employing a radical approximation. One assumes that the dissolution of each blob occurs after it has traveled a distance $\ell(r)$, known as the mixing length. (The mixing length at radius r is usually taken to be of the same order of magnitude as the scale height of the stellar fluid at that position, the radial distance in which density, pressure, etc. change appreciably, but it is difficult to justify this guess, and even more difficult to do better.) We assume that the whole mass of the star is involved in this convection, so the energy per time transported by convection through a sphere of radius r is the quantity (234) (with δr replaced with ℓ) times the mass $4\imath r^2\rho(r)\ell(r)$ in a shell of thickness $\ell(r)$ divided by the time $\approx \ell(r)/u(r)$ that it takes blobs to pass through this shell,

$$\mathcal{L}_{conv} \approx 4\pi r^2 u(r)\frac{\mathcal{E}(r)}{\rho(r)}\left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right)\ell(r) \tag{240}$$

where $u(r)$ is a typical blob velocity. To estimate u(r), we note that the buoyant force on a blob of volume V is the acceleration of gravity

$g = GM/r^2 = |p\prime/\rho|$ times the mass ρV of the ambient material with the same volume V minus the mass $(\rho+\delta\rho)V$ of the blob. To first order the acceleration of the blob is this force divided by ρV, which after traveling a distance $\ell(r)$ is

$$a = \left|\frac{p'}{\rho^2(r)}\right| \left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right) \ell(r) \tag{241}$$

The average velocity over this time is then of the order or $u \approx \sqrt{a\ell}$:

$$u(r) \approx \left|\frac{p'}{\rho^2(r)}\right|^{1/2} \left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right)^{1/2} \ell(r). \tag{242}$$

Using this $u(r)$ value in (240) we get

$$\mathcal{L}_{conv} \approx 4\pi r^2 \left|\frac{p'}{\rho^2(r)}\right|^{1/2} \frac{\mathcal{E}(r)}{\rho(r)} \left(\rho'(r) - \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)\right)^{3/2} \ell^2(r). \tag{243}$$

If we incorporate the condition of adiabatic stability (219)

$$\mathcal{L}_{conv} \approx \mathcal{L}_0 \left(\nabla(r) - \nabla_{ad}(r)\right)^{3/2} \tag{244}$$

with

$$\mathcal{L}_0 \equiv 4\pi r^2 \, \frac{p'^2(r)\mathcal{E}(r)\ell^2(r)}{p^{3/2}(r)\rho^{1/2}(r)}. \tag{245}$$

The convection will be efficient at r if the coefficient $\mathcal{L}_0$ is much larger than the luminosity L. This is often the case. If the mixing length $\ell^2(r)$ is half the pressure scale height, and $\mathcal{E} = (3/2)p$, we have

$$\mathcal{L}_0 \approx (3/2)\pi r^2 \, p^{3/2}(r)/\rho^{1/2}(r). \tag{246}$$

In the Sun at $r = 0.8R_\odot = 5.610^{10}cm$, $p = 1.610^{12}$ dyne/cm^2, and $\rho = 0.018$ g/cm^3, so $L_0 \approx 210^{41}$ erg/sec, as compared with the solar luminosity $L = 3.910^{33}$ erg/sec. This is the region of convection in the Sun and clearly due to the 8 orders of magnitude difference, very efficient.

In general cases of efficient convection, Eq. (244) requires that $\nabla(r)$ is very close to the adiabatic value $\nabla_{ad}(r)$. In particular, where $\mathcal{E}$ is related to the pressure by $\mathcal{E} = p/(\Gamma - 1)$, $\nabla_{ad}(r) = 1 - 1/\Gamma$, and so in the case of efficient convection we have

$$p(r) = K\rho^\Gamma(r), \tag{247}$$

where K is a constant that depends on conditions at the boundary of this region. This is the case throughout the convective region of the Sun, aside from a thin shell near the surface, where the pressure drops from 10^6 dyne/cm2 to 10^5 dyne/cm2. (a region where due to the effect of ionization, Γ is not constant, particularly in its outer parts.) Where Eq. (247) holds throughout a star's interior, the star is known as a polytrope. Another way of describing this process. According to the second law of thermodynamics there is a function S of ρ and p typically called entropy of the system, or specific entropy s –entropy per gram– in this case, that obeys the following equation

$$Tds = d(\mathcal{E}/\rho) + pd(1\rho). \tag{248}$$

Eq. (222) implies then

$$\delta s = 0. \tag{249}$$

We need to remember that throughout our formalism we neglected heat conduction, which is generally a good approximation in stars. In regions where convection is efficient the specific entropy tends to a nearly uniform value to keep the convective energy transport consistent with the actual luminosity of the star. Stars with a uniform entropy per gram are said to be isentropic.

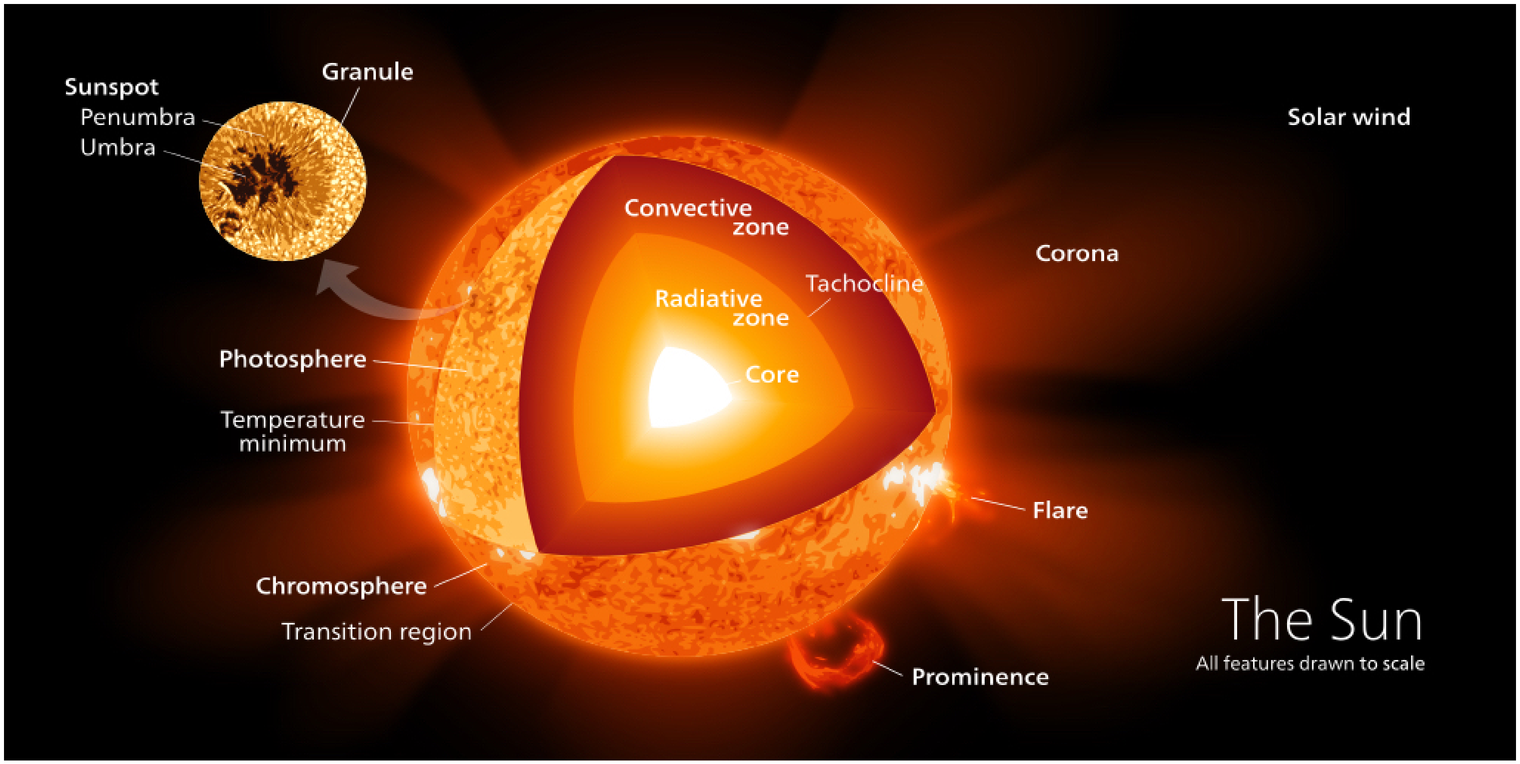

Figure 1: The different zones in the Sun.

If we use the polytropic equation

$$\mathcal{E} = \frac{p}{\Gamma - 1} \tag{250}$$

in (248) we get

$$\begin{aligned} T ds &= d\left(\frac{p}{(\Gamma-1)\rho}\right) + p d(1\rho) = \frac{1}{\Gamma-1}\left(\frac{dp}{\rho} + \Gamma p d\left(\frac{1}{\rho}\right)\right) \\ &= \frac{\rho^{\Gamma-1}}{\Gamma-1} d\left(\frac{p}{\rho^{\Gamma}}\right). \end{aligned} \tag{251}$$

Using the law of ideal gases in the form $T = p/R\rho$

$$ds = \frac{R}{\Gamma-1}\left(\frac{\rho^{\Gamma}}{p}\right) d\left(\frac{p}{\rho^{\Gamma}}\right) = \frac{R}{\Gamma-1} d\ln\left(\frac{p}{\rho^{\Gamma}}\right). \tag{252}$$

Then

$$s = \frac{R}{\Gamma-1}\ln\left(\frac{p}{\rho^{\Gamma}}\right) + \text{constant}. \tag{253}$$

If s is constant then p/ρ^{Γ} is also constant: i.e. $p\rho^{-\Gamma}$ is constant in an isentropic star. In regular main sequence stars there are regions of stability not affected by convection For these ones energy transport is by radiation and $p\rho^{-\Gamma}$ increases with r. In others there is a region of effective convection, in which $p\rho^{-\Gamma}$ is constant. For instance, in the Sun there is a core with radiative energy transport, extending from the center where $p \approx 210^{17}$ dyne/cm2, out to a radius about $0.65 R_{\odot}$ where the pressure has dropped to about 310^{13} dyne/cm2. Surrounding this region there is an outer convective layer, and (since convection cannot carry energy into empty space) a relatively thin surface layer dominated by radiative energy transport.
In more massive stars, there typically is a convective core, and an outer layer dominated by radiative transport that is stable against convection.

None of this affects the general results we just obtained because the radii where regions of convective energy transport begin or end, and the values of $p\rho^{-\Gamma}$ in these regions, are set by the conditions in the adjacent regions of radiative energy transport, and so are ultimately determined in terms of physical constants and the value of the nominal stellar radius R where the boundary conditions are $\rho(R) = p(R) = 0$.

In the case of isentropic stars (polytropes or not) the equations of hydrostatic equilibrium can be expressed as a variational principle. We can consider the variation in the total energy

$$E = \int_0^R 4\pi r^2 \left(\mathcal{E}(r) - \frac{G\mathcal{M}(r)\rho(r)}{r} \right) dr. \tag{254}$$

In terms of $\delta\rho$ and $\delta\mathcal{E}$ we have

$$\delta E = \int_0^R 4\pi r^2 \left(\delta\mathcal{E}(r) - \frac{G\mathcal{M}(r)\delta\rho(r)}{r} - \frac{G\rho(r)}{r} \int_0^r 4\pi r'^2 \delta\rho(r') dr' \right) dr. \tag{255}$$

Using that in the absence of heat flow the conservation of energy can be written

$$\delta(\mathcal{E}/\rho) + p\delta(1/\rho) = 0$$

We get that

$$\delta\mathcal{E} = (\mathcal{E} + p)\frac{\delta(\rho)}{\rho}$$

And using it in (255)

$$\delta E = \int_0^R 4\pi r^2 \left(\frac{\mathcal{E}(r) + p(r)}{\rho(r)} - \frac{G\mathcal{M}(r)}{r} - G\int_r^R 4\pi r' \rho(r') dr' \right) \delta\rho(r) dr. \tag{256}$$

And calling

$$\frac{\mathcal{E}(r) + p(r)}{\rho(r)} - \frac{G\mathcal{M}(r)}{r} - G\int_r^R 4\pi r' \rho(r') dr' = \mathcal{F}. \tag{257}$$

We can now calculate $\frac{d\mathcal{F}}{dr}$ using again $\delta(\mathcal{E}/\rho) + p\delta(1/\rho) = 0$

$$\frac{d\mathcal{F}(r)}{dr} = \frac{1}{\rho(r)}\frac{dp(r)}{dr} + \frac{G\mathcal{M}(r)}{r^2}. \tag{258}$$

But the fundamental equation of hydrostatic equilibrium for stars is

$$\frac{dp(r)}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}.$$

Implying that $\mathcal{F}(r)$ is a constant, i.e. $\mathcal{F}_0(r)$ and then (256) becomes

$$\delta E = \mathcal{F}_0(r)\int_0^R 4\pi r^2 \delta\rho(r) dr = \mathcal{F}_0(r)\delta\mathcal{M}. \tag{259}$$

In consequence the equation of hydrostatic equilibrium does not prescribe either a stationary E or M but it tell us that E is stationary if M is.

Polytropic stars

These are

$$p = K\rho^{\Gamma} \tag{260}$$

with ρ and Γ constant throughout the star.

Ordinary stars with efficient convective energy transport. As we just saw, these stars obey Eq. (260), with Γ typically close to 5/3, and K depending on boundary conditions, such as the values of the central density and pressure.

Also exceptionally light white dwarf stars obey Eq. (260) with $\Gamma \approx 5/3$, and exceptionally heavy white dwarf stars obey Eq. (260) with $\Gamma \approx 4/3$. In both cases K depends only on the chemical composition, as well as on fundamental physical constants.

Supermassive stars. These stars obey Eq. (260) with $\Gamma \approx 4/3$ and with K depending on the molecular weight and on the ratio of matter to radiation pressure, as well as on fundamental physical constants.

Let's look into polytropes without worrying too much about the reason for these stars obeying this particular relationship. Eq (260) does not depend on the temperature so we could concentrate on the hydrostatic equations (6) and (7)

$$\frac{dP(r)}{dr} = -G\frac{\rho(r)\mathcal{M}(r)}{r^2} \tag{6}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r) \tag{7}$$

We will rewrite these two first-order differential equations as a single second-order equation for the density:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{d}{dr}\rho^{\Gamma}(r)\right) + \frac{4\pi G}{K}r^2\rho(r) = 0. \tag{261}$$

We can take the central density to have some value $\rho(0)$ and its derivative also to be $\rho'(0) = 0$. This determines the solution up to Γ and K.

In the case of stars with efficient convective energy transport then there is not just one free stellar parameter, such as the star's mass or radius, but two free parameters, which can be taken as $\rho(0)$ and $K = p(0)/\rho^{\Gamma}(0)$. But this means that these stars do not obey the Vogt–Russell theorem, which asserts that for a definite chemical composition there is a unique solution to the equations of stellar structure, that depends on just a single stellar parameter, such as the radius R or the total mass M.

What happens in reality is that picking $\rho(R) = 0$ we are imposing $p(R) = 0$ at the surface due to the polytropic relationship $p(r)/\rho^{\Gamma} \propto$ constant. Having three first-order differential equations and only two independent parameter-free boundary conditions depending on R, there is an additional free parameter, which can be taken as K or $\rho(0)$, in addition to the radius R at which one of the boundary conditions is imposed.

The free parameters in eq (261) can be eliminated by rescaling the independent and dependent variables. Let's define

$$\Theta \equiv \left(\frac{\rho(r)}{\rho(0)}\right)^{\Gamma-1}. \tag{262}$$

Then (261) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\Theta\right) + \frac{4\pi G(\Gamma-1)}{K\Gamma}\rho^{(2-\Gamma)}(0)\Theta^{1/(\Gamma-1)} = 0. \tag{263}$$

defining

$$\xi \equiv \left(\frac{4\pi G(\Gamma-1)}{K\Gamma}\right)^{1/2}\rho^{(2-\Gamma)/2}(0)r. \tag{264}$$

(261) becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right) + \Theta(\xi)^{1/(\Gamma-1)} = 0, \tag{265}$$

which is called the Lane-Endem equation. Notice that this an essentially an equation for $\rho(r)$. Due to the prescribed relationship with p is also an equation for p. And if we use the equation of ideal gases there is also a relationship with T. we notice that

$$\Theta(0) = 1, \tag{266}$$

This is just the result of the definition of Θ in (262). Remember that $\xi = 0$ it also implies $r = 0$.

The boundary conditions can be obtained noticing the need to avoid a singularity at $\xi = 0$ in (265). To see this let's expand the derivative

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right) + \Theta(\xi)^{1/(\Gamma-1)} = \frac{2}{\xi}\frac{d\Theta}{d\xi} + \frac{d^2\Theta}{d\xi^2} + \Theta(\xi)^{1/(\Gamma-1)} = 0, \tag{267}$$

But $\frac{d\Theta}{d\xi}$ behaves essentially like $d\rho/dr$. From the equation of hydrostatic equilibrium applied to a small volume of infinitesimal radius δ so that the mass insides is $M_r = (4/3)\pi\overline{\rho}\delta^3$ we get

$$\frac{dP}{dr}(0) = -G\frac{M_r\rho}{r^2} = -\frac{4\pi}{3}G\overline{\rho}^2\delta \tag{268}$$

We see that $dp/dr \to 0$ when $\delta \to 0$. Since $p \propto \delta$ we also get $d\rho/dr \to 0$ when $\delta \to 0$.

$$\frac{d\Theta}{d\xi}(0) = 0. \tag{269}$$

This is the first boundary condition; the other one can be obtained from assuming that if a solution exist there should be a value 0 of the density ρ for a finite value of the radius which is the surface of the star. Let's have this radius determine a value $\xi = \xi_1$ such that

$$\Theta(\xi_1) = 0, \tag{270}$$

This radius would be

$$R = \left(\frac{4\pi G(\Gamma - 1)}{K\Gamma}\right)^{-1/2} \rho^{-(2-\Gamma)/2}(0)\xi_1. \tag{271}$$

Consequently the star's mass will be given

$$\begin{aligned} M &= \int_0^R 4\pi r^2\rho(r)dr \\ &= 4\pi\rho(0)^{(3\Gamma-4)2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\int_0^{\xi_1}\xi^2\Theta^{1/(\Gamma-1)}(\xi)d\xi. \end{aligned} \tag{272}$$

Using (265) the integral becomes

$$M = 4\pi\rho(0)^{(3\Gamma-4)2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\int_0^{\xi_1}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right)d\xi. \tag{273}$$

which gives

$$M = 4\pi\rho(0)^{(3\Gamma-4)2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\left(\xi^2\frac{d}{d\xi}\Theta(\xi)\right)\Bigg|_0^{\xi_1}$$
$$= 4\pi\rho(0)^{(3\Gamma-4)2}\left(\frac{K\Gamma}{4\pi G(\Gamma-1)}\right)^{3/2}\left({\xi_1}^2\frac{d}{d\xi}\Theta(\xi_1)\right). \quad (274)$$

There are three values of Γ for which exact non-singular solutions of the Lane-Emdem are known.

When $\Gamma = \infty$ (corresponds to $n = 0$), which essentially implies a constant density, we can propose a solution

$$\Theta(\xi) = a\xi^2 + c \quad (275)$$

When plug in (265) we get

$$\Theta(\xi) = -\frac{1}{6}\xi^2 + 1 \quad (276)$$

If we want to find ξ_1 we get

$$\Theta(\xi_1) = -\frac{1}{6}{\xi_1}^2 + 1 = 0 \quad (277)$$

as per (270) giving $\xi_1 = \sqrt{6}$ and we also get $\xi^2 d\Theta/d\xi(\xi_1) = -2\sqrt{6}$.

When $\Gamma = 2$ (corresponds to $n = 1$), (267) becomes

$$\frac{2}{\xi}\frac{d\Theta}{d\xi} + \frac{d^2\Theta}{d\xi^2} + \Theta(\xi) = 0, \quad (278)$$

For which the general solution is any linear combination of $\sin\xi/\xi$ and $\cos\xi/\xi$. The condition $\Theta(0) = 1$ gives the solution $\Theta(\xi) = \sin\xi/\xi$. This gives $\xi_1 = \pi$ and $\xi^2 d\Theta/d\xi(\xi_1) = -\pi$.

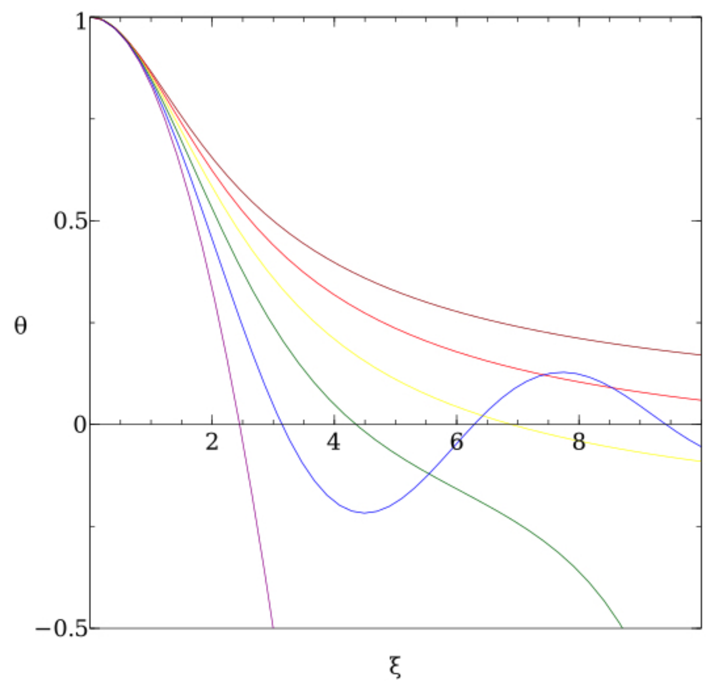

Figure 2: Θ as function of ξ for n=0, 1, 2, 3, 4 and 5 or Γ=∞, 2, 3/2, 4/3, 5/4, and 6/5.

For $\Gamma = 6/5$ the solution is

$$\Theta(\xi) = (1 + \xi^2/3)^{-1/2}. \tag{279}$$

Which means that the density is 0 only at infinity, i.e. a star with an infinite radius ($\xi = \infty$). But $\xi^2 \frac{d\Theta}{d\xi}(\xi) \to -\sqrt{3}$ when $\xi \to \infty$, which means using (274) that the mass is finite. Figure 2 shows Θ for different values of Γ.