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Instabilities

Note: these part of the notes are based on the book by Matt Benacquista, An Introduction to the Evolution of Single and Binary Stars and the lecture notes from Prof. Gary Glatzmaier from his notes on the Physics of Stars at UCSC

Let's look at the conditions that bring stability to processes in the star, to better understand how to treat instabilities after that.

The total energy of a star in hydrostatic equilibrium is the sum of the internal energy and the gravitational potential energy. Using the virial theorem, eq (25) and $d\mathcal{M} = 4\pi r^2 \rho dr$, the gravitational potential energy can be put in this form

$$\Omega = -3\int_0^R 4\pi r^2 P(r)dr = -3\int_0^M \frac{P}{\rho}d\mathcal{M}. \tag{280}$$

If the gas can be considered an ideal gas and there is a non-zero radiation pressure we have using the ideal gas law and eqs (120) and (121) that

$$\frac{P}{\rho} = \frac{P_{gas}}{\rho} + \frac{P_{rad}}{\rho} = \frac{R}{\mu}T + \frac{aT^4}{3\rho} = \frac{2}{3}u_{gas} + \frac{1}{3}u_{rad} \tag{281}$$

Integrating over all the mass we get

$$\int_0^M \frac{P}{\rho}d\mathcal{M} = \frac{2}{3}U_{gas} + \frac{1}{3}U_{rad} \tag{282}$$

Then

$$-\Omega = 2U_{gas} + U_{rad} \tag{283}$$

and

$$U_{gas} = -\frac{1}{2}(\Omega + U_{rad}) \tag{284}$$

The total energy is then

$$\mathcal{E} = U_{gas} + U_{rad} + \Omega = \frac{1}{2}(\Omega + U_{rad}) = -U_{gas} \tag{285}$$

Additionally we have that

$$\frac{d\mathcal{E}}{dt} = \dot{\mathcal{E}} = \mathcal{L}_{nuc} - L \tag{286}$$

This means that if the energy generated through nuclear fusion is not equal to the power radiated from the surface, the total energy will change. If we combine (285) and (286) we have

$$\mathcal{L}_{nuc} - L = -\dot{U}_{gas} \tag{287}$$

In equilibrium $\dot{\mathcal{E}} = 0$ and $\mathcal{L}_{nuc} = L$ This equation indicates that the internal energy of the star will drop if $\mathcal{L}_{nuc} > L$ or increase if $\mathcal{L}_{nuc} < L$.

Following eq (287) if $\mathcal{L}_{nuc} > L$ then $\dot{U}_{gas} < 0$, which implies that the internal energy of the gas will decrease and temperature will then decrease, leading itself to a decrease of the radiation energy, so $\dot{U}_{rad} < 0$ as well. But we also have that with $\mathcal{L}_{nuc} > L$ $\dot{\mathcal{E}} > 0$ so the total energy must increase. But

$$\dot{\Omega} = \dot{\mathcal{E}} - \dot{U} \tag{288}$$

And then the gravitational potential energy of the star must increase. Then the star will expand, and the density will decrease, and the temperature will decrease too. The nuclear reaction rate is from eq (175)

$$\epsilon(\rho, T) \simeq \epsilon_1 \rho^{\lambda} (k_B T)^{\nu}, \tag{289}$$

and consequently $\epsilon(\rho, T)$ will also decrease. And

$$\mathcal{L}_{nuc} = \int_0^M \epsilon dm \tag{290}$$

the nuclear generation rate will also decrease restoring equilibrium. A similar reasoning can be made in the case $\mathcal{L}_{nuc} < L$. The nuclear reaction rate will increase and equilibrium will be also restored.

Thermal Instability

An equivalente explanation of the previous situation can be constructed using the ideal gas law. An increase in $\mathcal{L}_{nuc}$ leads to an increase in temperature. This causes an increase in pressure and the gas expands and cools lowering the temperature and returning $\mathcal{L}_{nuc}$ to equilibrium. If the gas is not ideal, i.e. it is degenerate we have a different situation.

Degenerate matter is a highly dense state of fermionic matter in which the Pauli exclusion principle exerts significant pressure in addition to, or in place of thermal pressure. The description applies to matter composed of electrons, protons, neutrons or other fermions. The term is mainly used in astrophysics to refer to dense stellar objects where gravitational pressure is so extreme that quantum mechanical effects are significant. This type of matter is naturally found in stars in their final evolutionary states, such as white dwarfs and neutron stars, where thermal pressure alone is not enough to avoid gravitational collapse. Degenerate matter is usually modeled as an ideal Fermi gas, an ensemble of non-interacting fermions. In a quantum mechanical description, particles limited to a finite volume may take only a discrete set of energies, called quantum states. The Pauli exclusion principle prevents identical fermions from occupying the same quantum state. At lowest total energy (when the thermal energy of the particles is negligible), all the lowest energy quantum states are filled. This state is referred to as full degeneracy. This degeneracy pressure remains non-zero even at absolute zero temperature. Adding particles or reducing the volume forces the particles into higher-energy quantum states. In this situation, a compression force is required, and is made manifest as a resisting pressure. The key feature is that this degeneracy pressure does not depend on the temperature but only on the density of the fermions. Degeneracy pressure keeps dense stars in equilibrium, independent of the thermal structure of the star. A degenerate mass whose fermions have velocities close to the speed of light (particle energy larger than its rest mass energy) is called relativistic degenerate matter. The concept of degenerate stars, stellar objects composed of degenerate matter, was originally developed in a joint effort between Arthur Eddington, Ralph Fowler and Arthur Milne. Eddington had suggested that the atoms in Sirius B were almost completely ionized and closely packed.

In these cases $P_{gas} \propto P_e = K\rho^{\alpha}$ where $\alpha = 5/3$ for a Fermi gas (An ideal Fermi gas is a state of matter which is an ensemble of many non-interacting fermions –particles that obey Fermi–Dirac statistics–, like electrons, protons, and neutrons, and, in general, particles with half-integer spin. These statistics determine the energy distribution of fermions in a Fermi gas in thermal equilibrium, and is characterized by their number density, temperature, and the set of available energy states. The model is named after the Italian physicist Enrico Fermi.) In the case of an ultra relativistic gas $\alpha = 4/3$. In these cases an increase of $\mathcal{L}_{nuc}$ will not increase the pressure, in spite of increasing the temperature.

This is what happens: let's assume that in a time dt a shell of radius r is displaced to $r + dr = r(1 + xdt)$, where x is a constant. All the layers will increase proportionally.

$$\frac{\dot{r}}{r} = \frac{\partial \ln r}{\partial t} = x \tag{291}$$

Being x constant throughout the star

$$\frac{\partial}{\partial m}\left(\frac{\partial \ln r}{\partial t}\right) = 0 \tag{292}$$

which is also

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln r}{\partial m}\right) = \frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial m}\right) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi r^3 \rho}\right) = \frac{1}{4\pi r^3 \rho}\left(-3\frac{\dot{r}}{r} - \frac{\dot{\rho}}{\rho}\right) = 0 \tag{293}$$

$$-3\frac{\dot{r}}{r} = \frac{\dot{\rho}}{\rho} \tag{294}$$

Using the equation of hydrostatic equilibrium

$$P = \int_0^M \frac{Gm}{4\pi r^4} dm, \tag{295}$$

and taking the timederivative

$$\dot{P} = \int_0^M \frac{\partial}{\partial t}\left(\frac{1}{r^4}\right)\frac{Gm}{4\pi} dm = \int_0^M -\left(\frac{\dot{r}}{r}\right)\frac{Gm}{4\pi r^4} dm = -4\frac{\dot{r}}{r}P. \tag{296}$$

Combining (295) and (296) we get

$$\frac{\dot{P}}{P} = \frac{4}{3}\frac{\dot{\rho}}{\rho}, \tag{297}$$

which implies

$$\frac{dP}{P} = \frac{4}{3}\frac{d\rho}{\rho}. \tag{298}$$

We can take a look at gases with equation of state

$$P = K\rho^{\alpha}T^{\beta}. \tag{299}$$

Then

$$\frac{dP}{P} = \alpha\frac{d\rho}{\rho} + \beta\frac{dT}{T}. \tag{300}$$

which gives

$$\left(\frac{4}{3} - \alpha\right)\frac{d\rho}{\rho} = \beta\frac{dT}{T}. \tag{301}$$

If $\alpha < 4/3$ when the density increases the temperature will increase, as is the case with ideal gases where $\alpha = 1$. But for degenerate gases we saw that $\alpha \geqslant 4/3$ and also $\beta \ll 1$. The pressure is determined by ρ of electrons alone and its expansion will increase temperature very little. At the core of the star if the temperature of the star increases then $\mathcal{L}_{nuc}$ will also increase and then the density will decrease further, leading to an increase in temperature. But this is an unstable result that will continue until the temperature is so high and the density so low that the degeneracy is lifted.

Thin shell Instability

It sometimes occurs that nuclear burning takes place not in the entire star or its core, but in a thin shell within it. It typically happens in stars that have exhausted their hydrogen. The ashes from the heavier elements created during the fusion of H deposit at center of the star. The ash is supported by degeneracy pressure that the star is too cool to burn. On top of this ash there is still fuel left and burning continues. In this case the burning is generally confined to a thin shell sitting on top of the degenerate ash core. Consider a burning shell of mass $d\mathcal{M}$, temperature T , density ρ, outer radius r_{sh}, and inner radius r_0, which is taken to be fixed. The thickness is dr. The star is in hydrostatic equilibrium, so the pressure in the shell is determined by the equation of hydrostatic balance:

$$\frac{dP}{d\mathcal{M}} = -\frac{G\mathcal{M}}{4\pi r^4} \tag{302}$$

The pressure in the shell then is

$$P_{shell} = \int_{m_{shell}}^{M} \frac{G\mathcal{M}}{4\pi r^4} d\mathcal{M}, \tag{303}$$

where m_{shell} is the mass inside the shell.

Let's assume that the pressure changes a bit δP. If the outer boundary of the shell expands $1 + \delta r/r_{shell}$ then any shell of gas at a radius r moves to $r(1 + \delta r/r_{shell})$ after the perturbation. This is called an homologous expansion, a necessary condition for hydrostatic equilibrium.

And since this is the case, the pressure in the shell is now,

$$\begin{aligned} P_{shell} + \delta P &\approx \int_{m_{shell}}^{M} \frac{G\mathcal{M}}{4\pi[r(1+\delta r/r_{shell})]^4} d\mathcal{M} \\ &= \left(1 + \frac{\delta r}{r_{shell}}\right)^{-4} \int_{m_{shell}}^{M} \frac{G\mathcal{M}}{4\pi r^4} d\mathcal{M}, \end{aligned} \tag{304}$$

If we assume small perturbations then $\delta r \ll r_{shell}$ and we can use a polynomial Taylor expansion to approximate the coefficient in front of the integral.

Reminder

$$(1+x)^n = 1 + nx + O(x^2).$$

$$\begin{aligned} P_{shell} + \delta P &= \left(1 - 4\frac{\delta r}{r_{shell}}\right) \int_{m_{shell}}^{M} \frac{G\mathcal{M}}{4\pi r^4} d\mathcal{M} \\ \delta P &= -4\frac{\delta r}{r_{shell}} \int_{m_{shell}}^{M} \frac{G\mathcal{M}}{4\pi r^4} d\mathcal{M} \\ \frac{\delta P}{P} &= -4\frac{\delta r}{r_{shell}}. \end{aligned} \tag{305}$$

We can now look at the density change. The density in the shell is

$$\rho = \frac{d\mathcal{M}}{4\pi r_{shell}^2 dr}$$

After the perturbation the new density

$$\rho + \delta\rho = \frac{d\mathcal{M}}{4\pi r_{shell}^2 (dr + \delta r)} = \frac{d\mathcal{M}}{4\pi r_{shell}^2 dr}\left(1 + \frac{\delta r}{dr}\right)^{-1} = \rho\left(1 + \frac{\delta r}{dr}\right)^{-1}$$

Expanding again and keeping only the first term

$$\rho + \delta\rho \approx \rho\left(1 - \frac{\delta r}{dr}\right)$$

From where we obtain

$$\frac{\delta\rho}{\rho} = -\frac{\delta r}{dr} = -\frac{\delta r}{r_{shell}}\frac{r_{shell}}{dr} \tag{306}$$

Using (305) in (306) we get

$$\frac{dP}{P_{shell}} = 4\frac{dr}{r_{shell}}\frac{d\rho}{\rho}. \tag{307}$$

$\frac{dr}{r_{shell}}$ is a truly small number if we have a thin shell. This means that, for a given fractional change in density $\delta\rho/\rho$, the fractional change in pressure dP/P, is much smaller. We can look now at an equation of state for the gas of the shape $P \propto \rho^\alpha T^\beta$. We then have,

$$\frac{dP}{P} = \alpha\frac{d\rho}{\rho} + \beta\frac{dT}{T}. \tag{308}$$

Using the result from equation (307) in the latter one we get

$$\left(4\frac{dr}{r_{shell}} - \alpha\right)\frac{d\rho}{\rho} = \beta\frac{dT}{T}. \tag{309}$$

Instability occurs when the two coefficients have opposite signs. In this case the expansion of the star, which lowers ρ_c, increases the temperature,

increasing the nuclear reaction rate and more expansion. On the other hand pressure never decreases with increasing temperature, i.e. β is never negative. The coefficient on the left side of the equation, can be: β is always positive (1 for an ideal gas, $4/3$ for a degenerate relativistic gas, $5/3$ for a degenerate non-relativistic gas), and dr/r_{shell} can be arbitrarily small for a sufficiently thin shell.
Thin shells are always unstable to thermonuclear runaway. As with the case of a degenerate star, this runaway cannot continue indefinitely. We had that in the case of the degenerate star, stability returns when the temperature becomes high enough to remove the degeneracy. For a thin shell, stability returns when the pressure in the shell becomes large enough to expand it to the point where it is no longer thin, and dr/r_{shell} becomes larger than $\beta/4$.

Global Dynamical Stability

The instabilities studied so far involve situations in which hydrostatic equilibrium holds. Let's take a look into dynamical instabilities, in which hydrostatic balance disappears. The thermal time scale or Kelvin-Helmholtz time scale is the approximate time it takes for a star to radiate away its total kinetic energy content at its current luminosity rate, i.e.

$$\tau_{KH} \approx \frac{GM^2}{2RL}. \tag{310}$$

we will use the fact that $\tau_{KH} \gg \tau_{dyn}$. In a situation like this we can treat stars as adiabatic, i.e. neglect the heat exchange.

I. Stability Against Homologous Perturbations

A homologous perturbation is one in which we expand or contract the star uniformly, so that every shell expands or contracts by the same factor.

The equation of motion is

$$\ddot{r} = -\frac{G\mathcal{M}}{r^2} - \frac{1}{\rho}\frac{dP}{dr}. \tag{311}$$

Multiplying both sides by $d\mathcal{M} = 4\pi r^2 \rho dr$

$$d\mathcal{M}\ddot{r} = -\frac{G\mathcal{M}}{r^2}d\mathcal{M} - 4\pi r^2 dP. \tag{312}$$

where dP is the change in pressure across the shell.

Let's consider a star in equilibrium for which $\ddot{r} = 0$ for the shell and study a homologous perturbation with an expansion or contraction of the star $1 + \delta r$. The shell will move from radius r_0 to a radius $r_0(1 + \delta r/r_0)$. We assume a small perturbation $|\delta r/r_0| \ll 1$. $\delta r/r_0 > 0$ if there is an expansion or in the case of a contraction we have $\delta r/r_0 < 0$. The pressure itself will change from P_0 to $P_0(1+\delta P/P_0)$ also with $|\delta P/P_0| \ll 1$ as well.

The new unperturbed configuration will satisfy

$$0 = -\frac{G\mathcal{M}}{{r_0}^2}d\mathcal{M} - 4\pi {r_0}^2 dP_0. \tag{313}$$

If we insert the perturbed radius and pressure into eq. (312)

$$\begin{aligned} d\mathcal{M}\frac{d^2}{dt^2}\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right] = &- \frac{G\mathcal{M}}{[r_0(1+\delta r/r_0)]^2}d\mathcal{M} \\ &- 4\pi\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right]^2 d\left[P_0\left(1+\frac{\delta P}{P_0}\right)\right]. \end{aligned} \tag{314}$$

Expanding in a Taylor series and keeping terms linear in $\delta r/r_0$ and $\delta P/P_0$, and using that $-G\mathcal{M}d\mathcal{M}/{r_0}^2 + 4\pi {r_0}^2 dP_0 = 0$.

$$\begin{aligned} d\mathcal{M}\ddot{\delta r} &= -\left(1 - 2\frac{\delta r}{r_0}\right)\frac{G\mathcal{M}}{{r_0}^2}d\mathcal{M} - \left(1 + 2\frac{\delta r}{r_0} + \frac{\delta P}{P_0}\right)4\pi {r_0}^2 dP_0 \\ &= 2\frac{G\mathcal{M}}{{r_0}^3}\delta r d\mathcal{M} - 4\pi\left(2\frac{\delta r}{r_0} + \frac{\delta P}{P_0}\right){r_0}^2 dP_0, \end{aligned} \tag{315}$$

To be able to fully interpret this equation we need to add how P behaves as a function of r itself. We will use the fact that over short timescales, the star behaves adiabatically. So we have

$$P = K\rho^{\gamma}, \tag{316}$$

where γ is the adiabatic index which depends of the properties of the gas (degenerate or not, relativistic or not, etc). Suppose that the perturbation causes the density to change from its original value ρ_0 to a new value $\rho_0(1+\delta\rho/\rho_0)$. The perturbed pressure and density must satisfy the same adiabatic equation, so

$$\begin{aligned} P_0\left(1+\frac{\delta P}{P_0}\right) &= K\left[\rho\left(1+\frac{\delta\rho}{\rho_0}\right)\right]^{\gamma} \\ &\approx K{\rho_0}^{\gamma}\left(1+\gamma\frac{\delta\rho}{\rho_0}\right) \end{aligned} \tag{317}$$

From this we get

$$\delta P = K{\rho_0}^{\gamma}\gamma\frac{\delta\rho}{\rho_0} \tag{318}$$

From where we get

$$\frac{\delta P}{P_0} = \gamma\frac{\delta\rho}{\rho_0} \tag{319}$$

To finally relate this to r we use that for a shell its mass is given $d\mathcal{M} = 4\pi r^2\rho dr$ and then if we have a perturbation r and ρ will change homologously $r_0 \to r_0(1+\delta r/r_0)$, $dr_0 \to dr_0(1+\delta r/r_0)$ and $\rho \to \rho(1+\delta\rho/\rho_0)$ while the shell mass $d\mathcal{M}$ remains unchanged. Then we get

$$\begin{aligned} d\mathcal{M} &= 4\pi\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right]^2\rho_0\left(1+\frac{\delta\rho}{\rho_0}\right)dr_0\left(1+\frac{\delta r}{r_0}\right) \\ &= 4\pi{r_0}^2\rho_0 dr_0\left(1+3\frac{\delta r}{r_0}+\frac{\delta\rho}{\rho_0}\right), \end{aligned} \tag{320}$$

where, again, we have linearized and dropped higher-order terms in the perturbations. We also know that $d\mathcal{M} = 4\pi {r_0}^2 \rho_0 dr_0$ because the shell mass is unchanged. The parenthesis should be one implying that

$$\frac{\delta\rho}{\rho_0} = -3\frac{\delta r}{r_0} \tag{321}$$

Using this result in (320) we get

$$\frac{\delta P}{P_0} = -3\gamma\frac{\delta r}{r_0} \tag{322}$$

and now we can use the result in (316) and get

$$\begin{aligned} d\mathcal{M}\ddot{\delta r} &= 2\frac{G\mathcal{M}}{{r_0}^3}\delta r d\mathcal{M} - 4\pi {r_0}^2\left(2\frac{\delta r}{r_0} - 3\gamma\frac{\delta r}{r_0}\right) dP_0 \\ &= \left[2\frac{G\mathcal{M}}{{r_0}^2} d\mathcal{M} - 4\pi {r_0}^2 dP_0(2-3\gamma)\right]\frac{\delta r}{r_0}. \end{aligned} \tag{323}$$

Let's remember that the unperturbed configuration is

$$\frac{G\mathcal{M}}{{r_0}^2} d\mathcal{M} = -4\pi {r_0}^2 dP_0. \tag{324}$$

Using it in (323)

$$\ddot{\delta r} = -(3\gamma - 4)\frac{G\mathcal{M}}{{r_0}^3}\delta r. \tag{325}$$

which is the differential equation for a harmonic oscillator where the most general solution is $\delta r = Ae^{i\omega t}$, where

$$\omega = \pm\sqrt{(3\gamma - 4)\frac{G\mathcal{M}}{{r_0}^3}}. \tag{326}$$

For a non-relativistic ideal gas, $\gamma = 5/3$, so the term inside the square root is positive and ω is real. This means that $i\omega t$ is a pure imaginary number, so δr varies sinusoidally in time – the response of the star to the perturbation is to oscillate at a constant amplitude A. This is a stable behavior. On the other hand, suppose we have a gas that is not a non-relativistic ideal gas, and has a different value of the adiabatic index. If $\gamma < 4/3$, then the term inside the square root is negative, and ω is an imaginary number. In this case the term $i\omega t$ that appears in the numerator of the exponential is a real number, which can be positive or negative depending on whether we take the positive or negative square root – both are valid solutions. A negative real number corresponds to a perturbation that decays exponentially in time, which is stable. On the other hand, a positive real number for $i\omega$ corresponds to a solution for δr that grows exponentially in time. This is an instability, since it means that a small perturbation will grow to an arbitrary size, or to where our analysis in the limit of small δr no longer holds. The characteristic time required for this growth is just $1/(i\omega)$. Note that $1/(i\omega) \sim 1/\sqrt{G\rho} \sim t_{dyn}$. Thus if $\gamma < 4/3$, the star will be unstable on dynamical timescale.

II. Applications of Instability

The limit that a star becomes unstable for $\gamma < 4/3$ has consequences in a number of circumstances. A star that is dominated by the pressure of a relativistic gas (of either non-degenerate or degenerate electrons or of photons), approaches $\gamma = 4/3$. This causes stars that approach this limit to become unstable. Another situation where a star can approach $\gamma = 4/3$ is when ionization-type processes become important.

There are ionization-like mechanisms that operate at high temperatures which can also produce $\gamma < 4/3$, and these can have more severe consequences. One such example is the photodisintegration of iron nuclei and conversion of photons into electron-positron pairs at temperatures above several times 10^9 K. Both of these process are ionization-like in the sense that they used the increased thermal energy to create new particles rather than to make the existing particles move faster. Thus doing work on a gas in this condition does not cause its temperature to increase by any significant amount, and in doing work the temperature of the gas does not decrease much – everything is buffered by creation and destruction of particles. This is the hallmark of a gas with small γ. Unlike hydrogen ionization, these process can take place at the centers of stars and can involve a significant amount of mass. This analysis suggests that, if they do take place, they compromise the stability of the star as a whole. Indeed, this is exactly what it is thought to happen to initiate supernovae in massive stars: the core becomes hot enough that photodisintegration and/or pair creation push the adiabatic index below $4/3$, initiating a dynamical instability and collapse.

III. Opacity-Driven Instabilities

The mechanism for instabilities based on opacity, called the κ mechanism, was worked out the mid-20th century. However, the basic idea for opacity-driven instabilities was suggested by Eddington, based on analogy with a steam valve. Suppose there is a layer in a star that has the property that its opacity increases as it is compressed. If such a region is compressed, the increase in opacity will reduce the flow of heat through it, trapping more heat in the stellar interior. The layer acts like a valve that is closed. Closing the valve and trapping heat will raise the pressure interior to the opaque layer, causing it to expand. This expansion will decrease the opacity, opening the valve and letting the trapped heat out. This reduces the pressure in the stellar interior, reversing the expansion and letting the layer fall back. This raises its density and opacity, starting a new cycle. Clearly this mechanism only operates if the opacity increases with density. However, this is generally not the case. Free-free opacity obeys $\kappa \propto \rho T^{-7/2}$, so

$$\frac{d\kappa}{\kappa} = \frac{d\rho}{\rho} - \frac{7}{2}\frac{dT}{T}. \tag{327}$$

If the process is adiabatic

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho}, \tag{328}$$

and using the ideal gas law we also have

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{329}$$

and then

$$\frac{dT}{T} = (\gamma - 1)\frac{d\rho}{\rho}. \tag{330}$$

and back to (328)

$$\frac{d\kappa}{\kappa} = \left(\frac{9-7\gamma}{2}\right)\frac{d\rho}{\rho}. \tag{331}$$

Thus $d\kappa$ and $d\rho$ have opposite signs unless $\gamma < 9/7 = 1.29$. Since the star is unstable only if $d\kappa$ and $d\rho$ have the same sign, i.e. increasing density increases opacity, this means that instability occurs only for $\gamma < 9/7$.

Gasses composed of relativistic and non-relativistic, degenerate and non-degenerate particles all have $\gamma > 4/3$, so the instability does not operate throughout most of the star. However, we have just been reminded that γ can be small in the partially ionized zones of a star. In these regions of the star, this instability does operate, and these regions act like a piston, driving pulsations into the rest of the star. Whether these pulsations actually do anything significant depends on how large the instability zone is, where it is located in the star, and how luminous the star is. There are two main instability zones, one associated with hydrogen ionization and one with helium ionization. If the star is too hot, the ionization zones are located very close to the stellar surface, and thus they occur in a region where the density is low. This makes the piston ineffective, because it is driven by too little mass to excite motions in the rest of the star. Conversely, if a star is too cool, the ionization zones are deep in the star. The overlying layers of the star, which as we already saw are convective, then damp out the motions, and again nothing happens. Thus instability is possible only in a certain range of surface temperatures. Moreover, since the instability is ultimately driven by the star's radiation, so the strength with which it is driven depends on the star's luminosity. The instability does not operate if the luminosity is too low. It turns out that for this reason it does not generally operate in main sequence stars, because those which are luminous enough to meet the minimum luminosity condition are too hot at their surfaces, and those with cool enough surfaces are not sufficiently luminous. Post-main sequence stars, however, can be unstable to the κ mechanism, and this causes them to pulsate.

Stellar Pulsation and Variable Stars

The κ mechanism can cause instability in stars in several different parts of the HR diagram. The regions of instability are generally characterized by a minimum luminosity and a narrow range of surface temperatures, and thus are called **instability strips**, since they appear as vertical strips in the diagram. The most famous of the variable stars classes is the Cepheids.

The following is a list of some pulsating stars:

Type - I Cepheids (or classical Cepheids) are evolved stars with T_{eff} = 6000 to 8000 K found in the instability strip. They possess a high metallicity (these are young population - I stars) and periods in the range from approximately 1 to 100 days. Classical Cepheids were named after the prototype of this type of pulsating stars: the star δ Cephei.

Type - II Cepheids (or W Virginis stars) are evolved stars found in the instability strip. Typically their masses are lower than Type - I Cepheids

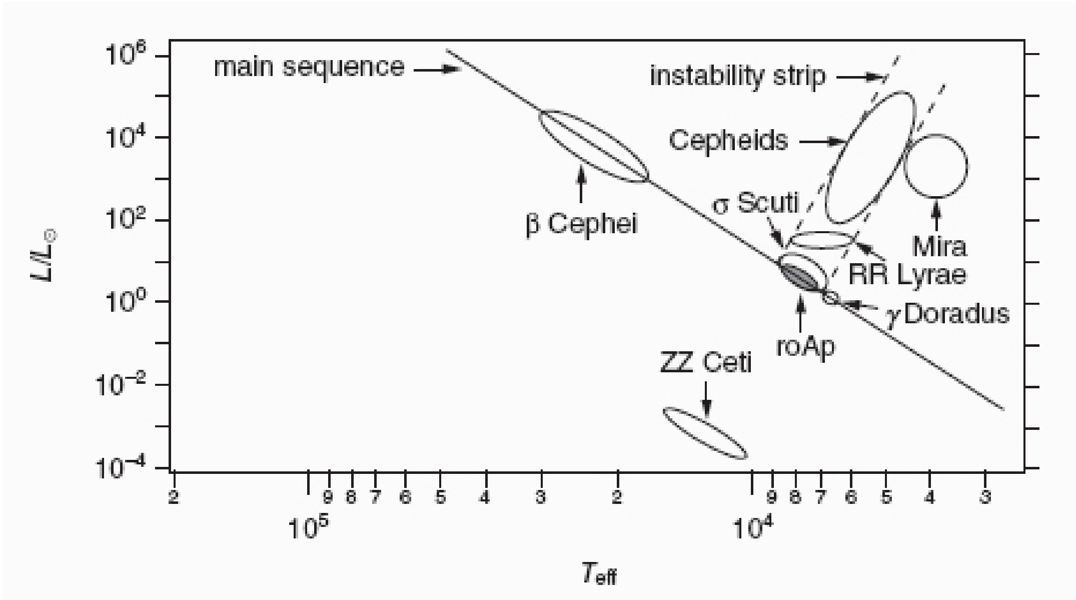

Figure 1: Position of pulsating/variable stars in the HR diagram. From F. Leblanc Introduction to Stellar Astrophysics

and are on the order of 0.5 M. They possess a low metallicity (these are old population - II stars) and periods from 1 to 50 days.

RR Lyrae stars are pulsating horizontal - branch stars with T_{eff} = 6000 to 7500 K. They are found in the instability strip of the H–R diagram. Their period of pulsation is typically in the range of 0.1 to 1 day. See the position fo these stars in figure 1.

Rapidly oscillating Ap stars (or $roAp$ stars) are magnetic Ap stars[1] on or near the main sequence that exhibit short periods of variability ranging from approximately 5 to 15 min. These stars are modeled by a star with a roughly dipolar magnetic field that is inclined with respect to its axis of rotation.

[1] Ap and Bp stars are chemically peculiar stars (hence the p) of types A and B which show overabundances of some metals, such as strontium, chromium and europium. In addition, larger overabundances are often seen in praseodymium and neodymium. These stars have a much slower rotation than normal for A and B-type stars, although some exhibit rotation velocities up to about 100 kilometers per second

β Cephei stars (or β Canis Majoris stars) are pulsating B- type stars on or slightly above the main sequence. Their periods of pulsations range from approximately 0.1 to 0.3 day. This class of stars is named after its prototype the star β Cephei. These stars should not to be confused with Cepheid stars.

Delta (δ) Scuti stars (or dwarf Cepheids) are pulsating A- or F - type stars on or near the main sequence with periods ranging from approximately 0.02 to 0.3 day. This class of stars is named after its prototype: the star δ Scuti.

Gamma γ Doradus stars are pulsating $F0$ - to $F2$ - type stars on or near the main sequence with periods ranging from approximately 0.4 to 3 days. This class of stars is named after its prototype: the star γ Doradus.

Mira variables are evolved (red giant) stars with periods ranging from approximately 100 to 1000 days. They are found just outside (on the cool side) of the instability strip. This class of stars is named after its prototype: the star Mira or Omicron Ceti. Figure2 shows the observed light curve of the star Omicron Ceti over a decade. Mira variables are part of a larger class of variable stars called long - period variables (LPVs).

ZZ Ceti stars are pulsating white dwarfs stars of spectral type DA . These may also be written DAV, V standing for variable. These stars exhibit photometric variations of up to approximately 0.3 magnitude with pulsating periods ranging from approximately 100 to 1000 s. White dwarfs from other spectral types can also pulsate (i.e. DBV, DOV and DQV) .

In addition to the list given above, there exist many other types of pulsating stars. For example, the Sun is a pulsating star. The Sun possesses a very large number of pulsation modes with characteristic periods on the order of 5 min. Solar pulsations have amplitudes that are much smaller

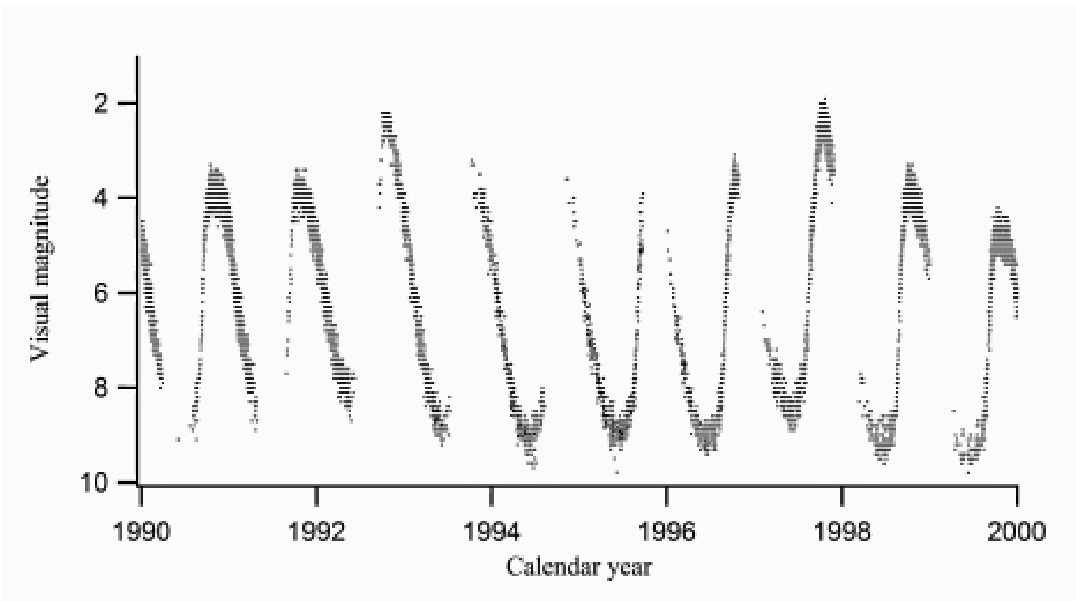

Figure 2: Observed visual magnitude for the prototype star for Omicron Ceti (a Mira star) from 1990 to 2000. This star has a period of approximately 332 days, $M \approx 0.7 M_{\odot}$ and $T_{eff} \approx 3000K$. Data courtesy of the American Association of Variable Star Observers (www.aavso.org). From F. Leblanc Introduction to Stellar Astrophysics

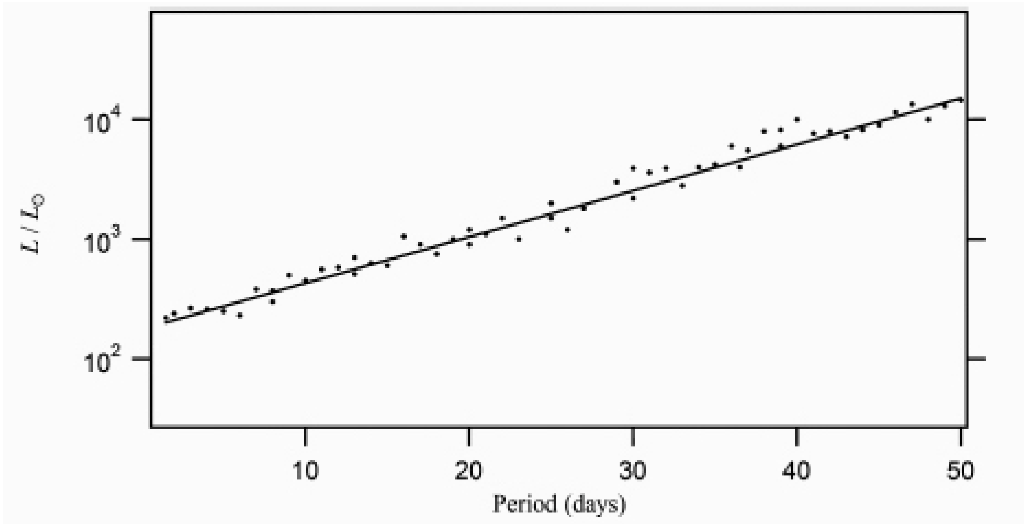

Figure 3: Illustration of the period–luminosity relation for classical (or Type - I) Cepheid stars. From F. Leblanc Introduction to Stellar Astrophysics

than those shown above for Mira and below for a classical Cepheid (see Figure 4) . Unlike these two types of pulsating stars, most stars do not have a dominant period of pulsation and a large number of pulsation modes can be present. Of course, the Sun also shows rotational (due to sunspots) and eruptive (solar flares) variability.

Variable stars are important because the period of the oscillation depends on the luminosity of the star – which is not surprising, since the luminosity determines how hard the instability is driven (see figure 3. This relation was first discovered empirically in 1908 by Henrietta Swan Leavitt, and has now been understood from first principles.

The Cepheid period-luminosity relation is important in astronomy because it provides a distance indicator. Since one can compute the luminosity from the star's observed period, one can determine its distance by comparing the observed heat flow to the luminosity. Cepheids are bright enough to be seen in other galaxies, and thus can be used to determine the distance to those galaxies. This technique was first used on a large number

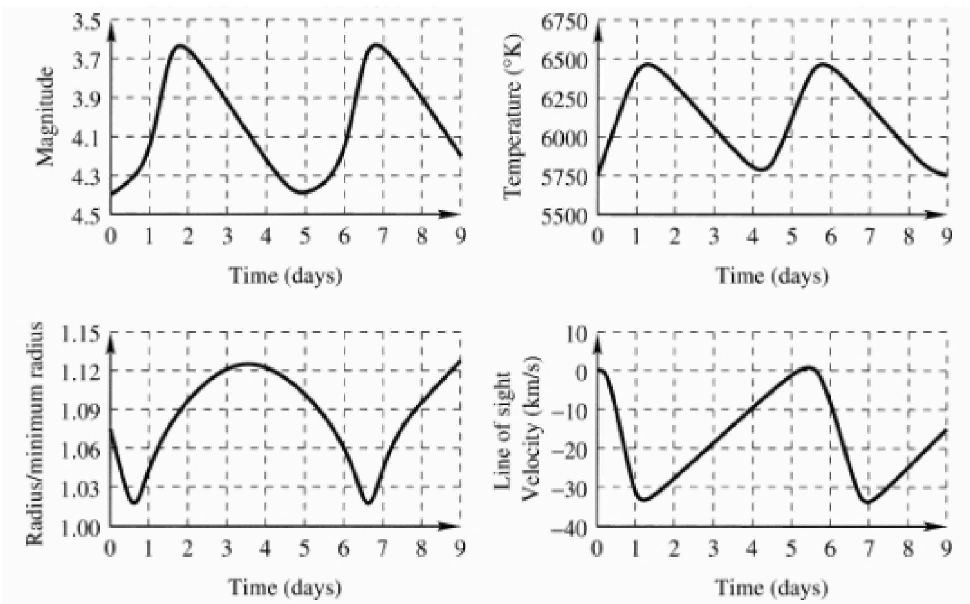

Figure 4: Magnitude, temperature, radius and line of sight velocity (or radial velocity) of its surface as a function of time for the classical Cepheid star δ Cephei. From F. Leblanc Introduction to Stellar Astrophysics

of galaxies by Edwin Hubble, leading to the discovery of the expansion of the universe.