Introduction to Astrophysics
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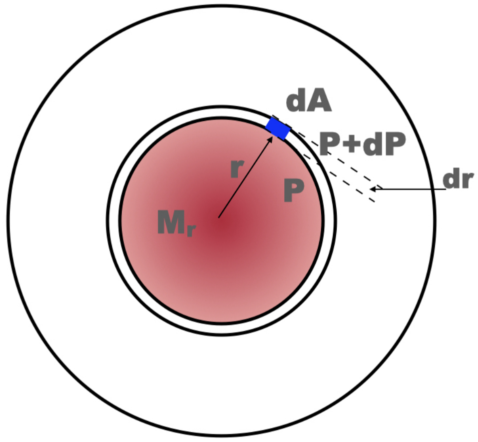

Figure 1: The internal pressure of the gas counterbalances the pull of gravity in a thin spherical shell of radius dr.

0.1 Lecture 1 Stars

Stars are huge balls of gas, many million times more massive than the Earth. We know they can radiate energy for millions of years. We also know that for geological eras spanning millions of Earth years our Sun has been pretty stable. Its energy has not change much. it has been in equilibrium. Let's examine Hydrostatic equilibrium first:

1.1 Hydrostatic equilibrium

Gravity pulls inward. In equilibrium the thermal motion of the gas molecules induce a pressure counterbalancing gravity. Let's find an expression for this equilibrium of the gas. If we consider a thin spherical shell of radius dr at a radius r from the center (see figure **??**), the gravitational force on a dA on the shell is:

$$
\begin{aligned}
dF_{Grav} &= -G\frac{\mathcal{M}(r)dm}{r^2} \\
&= -G\frac{4\pi r^2\rho(r)\mathcal{M}(r)drdA}{r^2} \\
&= -4\pi G\rho(r)\mathcal{M}(r)dr
\end{aligned} \tag{1}
$$

where

$$
\mathcal{M}(r) = \int_0^r 4\pi r'^2\rho(r')dr' \tag{2}
$$

The force due to the difference pressure at the piece of the shell dA is:

$$
dF_p = (P + dP)dA - PdA = dPdA \tag{3}
$$

Over the entire shell it will be

$$
dF_p = -dP\int_0^r dA = 4\pi r^2 dP \tag{4}
$$

In equilibrium, both forces are equal and we get,

$$
-4\pi G\rho(r)\mathcal{M}(r)dr = 4\pi r^2 dP(r) \tag{5}
$$

from where we obtain

$$
\frac{dP(r)}{dr} = -G\frac{\rho(r)\mathcal{M}(r)}{r^2} \tag{6}
$$

A very useful relationship to study the behavior of pressure as a function of the radius can be obtained from observing first that (2) can be cast as a differential relation as well:

$$
\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2\rho(r) \tag{7}
$$

and noticing that

$$\frac{d}{dr}\left(\frac{G\mathcal{M}^2(r)}{8\pi r^4}\right) = -\frac{G\mathcal{M}^2(r)}{2\pi r^5} + \frac{G\mathcal{M}(r)\mathcal{M}'(r)}{4\pi r^4} \tag{8}$$

where $\mathcal{M}'(r)$ is the derivative of $\mathcal{M}$ respect to r and it's given by (7). Then we can calculate:

$$\frac{d}{dr}\left(P(r) + \frac{G\mathcal{M}^2(r)}{8\pi r^4}\right) =$$
$$-\frac{G\mathcal{M}(r)\rho(r)}{r^2} - \frac{G\mathcal{M}^2(r)}{2\pi r^5} + \frac{G\mathcal{M}(r)\mathcal{M}'(r)}{4\pi r^4} \tag{9}$$

But precisely using (7) the first and third term in that equation are the same. And due to the fact that the second term is negative ($\frac{G\mathcal{M}^2(r)}{2\pi r^5}$ is positive) we get the following final inequality

$$\frac{d}{dr}\left(P(r) + \frac{G\mathcal{M}^2(r)}{8\pi r^4}\right) = -\frac{G\mathcal{M}^2(r)}{2\pi r^5} \leqslant 0 \tag{10}$$

We have actually prove a theorem:

Theorem I

In an equilibrium configuration the function $P + \frac{GM^2(r)}{8\pi r^4}$ decreases outward[1].

Corollary

Let's use it to calculate the density of a star when $r \to 0$. We can assume that the density is finite at $r = 0$. When $r \to 0$, $\mathcal{M}(r) \propto r^3$. Then

[1] S. Chandrasekhar, An introduction to the study of stellar structure (Dover, 1958).

the second term in (10) behaves $M^2(r)/r^4 \to 0$. The only remaining term is then $P(r=0) + \frac{G\mathcal{M}^2(r)}{8\pi r^4}|_{r=0} = P_c$.

At the star's surface we should have $P(r=R)=0$ and then $\mathcal{M}=M$ where M is the total mass of the star. Then using Theorem I we get then that:

$$P_c > P(r) + \frac{G\mathcal{M}^2(r)}{8\pi r^4} > \frac{GM^2}{8\pi r^4} \tag{11}$$

which gives us that:

$$P_c > \frac{GM^2}{8\pi r^4} \tag{12}$$

Exercise:

If we use the values for the mass and the radius of the sun we get:

$$\begin{aligned} P_c \geqslant \frac{GM^2}{2\pi R^4} &= 4.44 \times 10^{14} dynes/cm^2 \\ &= 4.50 \times 10^8 atmospheres \end{aligned} \tag{13}$$

Or for any other star in terms of the mass of the sun:

$$\begin{aligned} P_c \geqslant \frac{GM^2}{2\pi R^4} &= 4.44 \times 10^{14} \frac{(M/M_\odot)}{(R/R_\odot)^4} dynes/cm^2 \\ &= 4.50 \times 10^8 \frac{(M/M_\odot)}{(R/R_\odot)^4} atmospheres \end{aligned} \tag{14}$$

The Virial Theorem of Classical Mechanics

Definition refresher: *Homogeneous real-valued function*: of two variables x and y is a real-valued function that satisfies the condition $f(rx, ry) = r^k f(x,y)$

for some constant k and all real numbers r. The constant k is called the degree of homogeneity.

Euler's homogeneous function theorem
Suppose that the function $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is continuously differentiable. Then f is positively homogeneous of degree k if and only if $\mathbf{x} \cdot \nabla f(\mathbf{x}) = kf(\mathbf{x})$. [2]

If the potential energy is a homogeneous function of the coordinates (exercise: what degree of homogeneity is the potential energy in the Newtonian gravity; and the Coulomb force?) and the motion takes place in a finite region of space, there is a simple relationship between the average values of the kinetic and potential energies, which plays a prominent role in Astronomy, called the Virial Theorem[3].

If we apply the Euler's theorem to the kinetic energy of a physical system, due to the fact that $T = T(\vec{v}^2)$, we get $\sum\limits_a \mathbf{v}_a \cdot \partial T/\partial \mathbf{v}_a = 2T$ we can write:

$$2T = \sum_a \mathbf{p}_a \cdot \mathbf{v}_a = \frac{d}{dt}\left(\sum_a \mathbf{p}_a \cdot \mathbf{r}_a\right) - \sum_a \mathbf{r}_a \cdot \dot{\mathbf{p}}_a \tag{15}$$

We can now average this equation with respect to time. The average value of a function of time can be defined:

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t) dt$$

But if $f(t)$ is the time derivative $dF(t)/dt$ of a bounded function $F(t)$,

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{dF}{dt} dt = \lim_{\tau \to \infty} \frac{F(\tau) - F(0)}{\tau} = 0$$

[2] for a proof see https://en.wikipedia.org/wiki/Homogeneous_function.

[3] i.e. see L. Landau, Classical Mechanics.

If we now look at equation (15) $\sum\limits_a \mathbf{p}_a \cdot \mathbf{r}_a$ is bounded (positions remain within a given region and momenta are not infinite) so when we take the average of $2T$ the first term on the right hand side is 0. For the second term we can replace $\dot{\mathbf{p}}_a$ by $-\partial U/\partial \mathbf{r}_a$ following Newton's second law and get:

$$2\bar{T} = \overline{\sum_a \mathbf{r}_a \cdot \partial U/\partial \mathbf{r}_a} \tag{16}$$

Using now Euler's theorem:

$$2\bar{T} = k\bar{U} = -\bar{U} \tag{17}$$

due to the fact that in Newton's gravitational potential is $k = -1$. Since $\bar{T} + \bar{U} = \bar{E} = E$ (energy is conserved!), (17), can be expressed as:

$$\bar{U} = 2E \ , \ \bar{T} = -E \tag{18}$$

expressing $\bar{U}$ and $\bar{T}$ in terms of the total energy of the system. In Astronomy $\overline{\sum\limits_a \mathbf{r}_a \cdot \partial U/\partial \mathbf{r}_a}$ is called the *virial* of the system.