Introduction to Astrophysics
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Mario C Díaz

0.1 Lecture 1
Stars *Continuation*

Star Energy

At a distance $r > R$ where R is the radius of the star, $\mathcal{M}(r) = M$ where M is the total mass of the star, the potential energy is

$$V = -\frac{GM}{r} \qquad (r \geqslant R) \tag{19}$$

Where we assume $V = 0$ as $r \to \infty$. At the surface of the star we have

$$V = -\frac{GM}{R} \tag{20}$$

The potential energy of the star can be defined as the work done (on the star) to bring the matter from infinity into the actual distribution. In what follow we will call it Ω. It can be calculated as follows: Assume we have already "brought" from infinity $\mathcal{M}(r)$. The work necessary to bring an additional amount of matter $d\mathcal{M}(r)$ -as a spherical shell of the star matter- is

$$-G\mathcal{M}(r)d\mathcal{M}(r)\int_r^\infty \frac{dr}{r^2} = -\frac{G\mathcal{M}(r)d\mathcal{M}(r)}{r} \tag{21}$$

And consequently the gravitational potential energy of this configuration is:

$$-\Omega = G\int_0^R \frac{\mathcal{M}(r)d\mathcal{M}(r)}{r} \tag{22}$$

This equation is valid even if the star is not in equilibrium. But if it is we can use our results. But before let's use that from $\rho = \mathcal{M}(r)/V$ we obtain

$$d\mathcal{M}(r) = 4\pi r^2 \rho dr$$

and then (22) becomes:

$$-\Omega = 4\pi G \int_0^R r\mathcal{M}(r)\rho(r)dr \tag{23}$$

where R is the radius of the surface where $P(R) = 0$. Using (6) we get,

$$\Omega = 4\pi \int_0^R \frac{dP(r)}{dr} r^3 dr \tag{24}$$

Integrating by parts

$$\int_0^R \frac{dP(r)}{dr} r^3 dr = P(r)r^3|_0^R - \int_0^R 3Pr^2 dr$$

and so the first term cancels because is 0 at both limits of integration and the result is:

$$\Omega = -3 \int_0^R 4\pi P(r)r^2 dr \tag{25}$$

On the other hand we can also work out from (22) and using that $\mathcal{M}(r) = \frac{1}{2}\frac{d}{d\mathcal{M}}\mathcal{M}^2$

$$\begin{aligned} -\Omega &= \frac{1}{2} G \int_0^R \frac{d}{d\mathcal{M}}(\mathcal{M}^2)\frac{1}{r} d\mathcal{M}(r) \\ &= \frac{1}{2}\frac{GM^2}{R} + \frac{1}{2}\int_0^R \frac{\mathcal{M}^2}{r^2} dr \end{aligned} \tag{26}$$

were we used that $\mathcal{M}(r = R) = M$ and then using (19) we get that

$$\frac{dV}{dr} = \frac{G\mathcal{M}(r)}{r^2} \tag{27}$$

which inserted in (26) gives

$$-\Omega = \frac{1}{2}\frac{GM^2}{R} + \frac{1}{2}\int_0^R \frac{dV}{dr}\mathcal{M}(r)dr \tag{28}$$

Integrating once again by parts and using that the Newtonian potential at the boundary is $V = -\frac{GM}{r}$ we obtain

$$\Omega = \frac{1}{2}\int_0^R V(r)d\mathcal{M}(r) = \frac{1}{2}\int_0^R V(r)\frac{d\mathcal{M}}{dr}dr \tag{29}$$

The total energy of the star is the sum of Ω and the star's thermal energy, which is given by

$$\Upsilon \equiv \int_0^R \mathcal{E}(r)4\pi r^2 dr \tag{30}$$

where $\mathcal{E}$ is the density of internal thermal energy and it does not include the rest energy of the star or its potential energy. The total non-relativistic energy of the star is

$$E = \Upsilon + \Omega \equiv \int_0^R \left[\mathcal{E}(r) - 3P(r)\right] 4\pi r^2 dr \tag{31}$$

where we use the expression (25) for Ω. From this equation it is clear that the star has negative energy if $\mathcal{E}(r) < 3p(r)$.

When the energy is proportional to the pressure by convention it is used the following expression

$$\mathcal{E} = \frac{p}{\Gamma - 1} \tag{32}$$

When this is the case the stars are called polytropes.

(A polytropic process is a thermodynamic process that obeys the relation: $pV^n = C$ where p is the pressure, V is volume, n is the polytropic index.) Example: An ideal gas of monoatomic particles has $p = nk_BT$ and $\mathcal{E} = 3nk_BT/2$. In this case $\Gamma = 5/3$. In the case of radiation -a gas of photons- $p = \mathcal{E}/3$, and then $\Gamma = 4/3$.

In all these cases the thermal and gravitational energies of the star are given in terms of the total (non-relativistic) energy by equations (25), (30) and (32) as

$$\Upsilon = -\frac{E}{3\Gamma - 4}, \qquad \Omega = \frac{(\Gamma - 1)E}{\Gamma - \frac{4}{3}} \tag{33}$$

Stability (equilibrium) requires $E < 0$ (why?: if E is positive it means that the kinetic energy of the star is larger than the binding negative potential energy, and this means essentially that the star explodes). If $E < 0$, and considering that also $\Omega < 0$ we can see that stability requires $\Gamma > 4/3$. Very massive stars and also neutrons stars and white dwarfs with the maximum possible mass are dominated by highly relativistic matter. They are all on the edge of exploding. They have Γ barely above $4/3$.

Equation (33) determines the behavior of stars at a very early stage. The total energy E of a cold cloud of gas with low density will have little low gravitational energy (its molecules are loose and dispersed), so its total energy E will be small. But even at low temperatures this cloud will radiate at low, mainly infrared, wavelengths. If its total energy becomes negative, the cloud will no longer be able to disperse. According to Eq. (33), as the cloud loses energy then, as long as $\Gamma > 4/3$, Ω will decrease, becoming increasingly negative, but the internal energy Υ will increase. The star behaves as if it has negative specific heat; the more it loses energy, the hotter it gets. With increasing temperature the star radiates energy more rapidly, and the process accelerates. Eventually the central temperature of the star becomes so high that nuclei can penetrate the Coulomb repulsive barrier that separates them; nuclear energy generation begins and increases until it balances the energy lost by radiation; and the star becomes stable, at least until the star's center runs out of nuclear fuel. In a seemingly contradictory way, the onset of nuclear reactions stops the heating of the star. As a protostar radiates energy and heats up, it also contracts.
We can define a mass-weighted mean radius $\bar{r}$

$$\bar{r} \equiv \frac{M^2}{\int_0^R r\mathcal{M}(r)\rho(r)dr} \tag{34}$$

Notice that the quantity on the right hand side has dimensions of radius. With this definition equation (22) can be written $\Omega = -4\pi GM^2/\bar{r}$.

This is a neat result easy to picture: as $-\Omega$ increases for a given amount of total mass, $\bar{r}$ must decrease.

1.2 Radiative Energy Transport

In this section we will study the equations of energy transport, which rule how the temperature varies through the star. Conduction is negligible due to the fact that the mean free path of gas molecules in stars is low. Remember that:

$$\frac{1}{A}\frac{dQ}{dt} = -\kappa\nabla T \tag{35}$$

where A is the area through which the heat Q flows, t is the time, κ is the conductivity and ∇T the temperature gradient. The conductivity itself is defined

$$\kappa = \frac{n\bar{v}\lambda c_V}{3N_A} \tag{36}$$

where n is the number of particles per volume, $\bar{v}$ is the mean velocity of the molecules, λ is the mean free path of the particles, c_V is the molar heat capacity, and N_A is the Avogadro's number.

So convection and radiation are the main energy transport mechanisms within stars. Let $\ell(\hat{n}, \mathbf{x}, \nu, t)d^2\hat{n}d\nu$ be the energy per volume at position $\mathbf{x}$ and time t of photons with directions within a solid angle $d^2\hat{n}$ around the unit vector $\hat{n}$ and frequencies between ν and $\nu + d\nu$.

We will first calculate various contributions to the rate of change of $\ell(\hat{n}, \mathbf{x}, \nu, t) d^2\hat{n} d\nu$.

Transport

If the radiation is stationary then after a time $t + dt$ compared to a previous time t, the energy $\ell(\hat{n}, \mathbf{x}, \nu, t)$ will only change due to the position of the photons carrying it.

$$\ell(\hat{n}, \mathbf{x}, \nu, t + dt) = \ell(\hat{n}, \mathbf{x} - c\hat{n} dt, \nu, t) \tag{37}$$

and the temporal rate of change of ℓ due to transport of radiation is

$$\frac{\partial}{\partial t} \ell(\hat{n}, \mathbf{x}, \nu, t) = -c\hat{n} \cdot \nabla \ell(\hat{n}, \mathbf{x}, \nu, t) \tag{38}$$

Absorption

Absorption of a photon is a process in which the photon is absorbed, i.e. it disappears and no other photon correlated with the original one is emitted in the process. Examples of absorption are "bound-free" transitions, "free-free transitions" and "bound-bound" transitions.

"Bound-free" transitions occur when a photon is absorbed by a bound electron which increases its energy to become a free particle.

"Free-free transitions" occur when the photon is absorbed by a free electron which will also increase its temperature. This energized electron could raise the temperature of the medium and stimulate the emission of another photon. But this other one will be uncorrelated with the first one.

In a "bound-bound" transition the photon could raise the energy of a whole atom. This atom could undergo collisions but even if it losses energy emitting another photon this latter one will still be uncorrelated with the first one.

Let's assume that the net fraction of radiation of frequency ν absorbed at position $\mathbf{x}$ and time t in a time interval dt is $c\kappa_{abs}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)dt$, where ρ is the mass density and κ_{abs} is a coefficient called the absorption opacity, characteristic of the medium. (in the case of stimulated emission it counts as negative absorption.) (Notice that $c\kappa_{abs}\rho$ has a dimension of length and is the average distance that a typical photon travels before being absorbed in a homogeneous medium.)

Then the rate of change of ℓ due to absorption is

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{abs} = -c\kappa_{abs}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)\ell(\hat{n},\mathbf{x},\nu,t), \qquad (39)$$

$\kappa_{abs}\rho$ is the absorption cross section times the number density of absorbers for the bound-free or bound-bound transitions. We will see later that the free-free transitions it is a different situation.

Scattering

In these processes the disappearance of an initial photon yields a final photon, whose direction generally differs from the initial direction, but is correlated with it. the elastic scattering of photons with energies well below $m_e c^2$ on non-relativistic electrons, called Thomson scattering, is an example of such processes.

The fraction of energy radiated of frequency ν traveling in a direction $\hat{n}$ that during an interval dt at a time t which is scattered at the position $\mathbf{x}$ into a solid angle $d^2\hat{n}'$ around a final direction $\hat{n}'$, is written as $-c\kappa_s(\hat{n} \to \hat{n}'; \mathbf{x}, \nu, t)$ where κ_s is a coefficient characterizing the scattering particles, independent of the photon distribution function ℓ.

To calculate the rate of change of $\ell(\hat{n}, \mathbf{x}, \nu, t)$ we must now take into account the scattering of photons at position $\mathbf{x}$ and time t with initial directions $\hat{n}$ into any other directions $\hat{n}'$ and also the earlier scattering of photons any where else with also arbitrary initial directions $\hat{n}'$ into the position $\mathbf{x}$ and direction $\hat{n}$.

For this purpose, we assume that $1/\kappa_s\rho$ is so much smaller than the distance over which conditions in the star vary that we can assume that any photon that after scattering reaches a given position $\mathbf{x}$ at time t can only have been scattered at a position and time where the photon distribution function ℓ and density ρ were essentially the same as at $\mathbf{x}$ at t (something that might not be true at the surface).

The contribution of scattering to the rate of change of ℓ is:

$$\begin{aligned} \left(\frac{\partial}{\partial t}\ell(\hat{n}, \mathbf{x}, \nu, t)\right)_{scat} &= \\ -c\rho(\mathbf{x}, t) \int d^2\hat{n}' [-\kappa_s(\hat{n} \to \hat{n}'; \mathbf{x}, \nu, t)\ell(\hat{n}, \mathbf{x}, \nu, t) & \\ +\kappa_s(\hat{n}' \to \hat{n}; \mathbf{x}, \nu, t)\ell(\hat{n}', \mathbf{x}, \nu, t)] & \end{aligned} \tag{40}$$

We will ignore changes in frequency during scattering. This is valid if the photon energy is much lower than the rest energy of the particles with which they are colliding and the speeds are non relativistic. An exception is when resonant scattering occurs.

We have then

$$\kappa_s(\hat{n} \to \hat{n}'; \mathbf{x}, \nu, t) = N_{scat}(\mathbf{x}, t)\sigma(\hat{n} \to \hat{n}', \nu)$$

where $\sigma(\hat{n} \to \hat{n}', \nu)$ is called the differential scattering cross section, and $N_{scat}(\mathbf{x}, t)$ is the number of scattering particles (centers) per unit mass (typically gram).