Introduction to Astrophysics
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0.1 Lecture 1
Stars *Continuation*

Emission (thermal and nuclear)

We assume now that the radiation energy emitted in any direction per time, per volume, per solid angle, and per frequency interval at position $\mathbf{x}$ and time t is:

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{emitted} = j(\mathbf{x},\nu,t))\rho(\mathbf{x},t)/4\pi, \qquad (41)$$

where j is a coefficient specific to the medium and the radiation field (in some books called the emission coefficient and indicated j_λ or j_ν). It includes any radiation emitted isotropically after a photon absorption, in addition to the ordinary thermal radiation from the stellar material, which is heated by nuclear processes. (Stimulated emission, which creates a photon with the same momentum and helicity as one already present, will be included as a negative term in the absorption coefficient κ_{abs}.) Putting together these terms,

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right) = \left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{transport} + \left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{absorption} + \left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{scatt} + \left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right)_{emitted}$$

which gives:

$$\left(\frac{\partial}{\partial t}\ell(\hat{n},\mathbf{x},\nu,t)\right) = -c\hat{n}\cdot\nabla\ell(\hat{n},\mathbf{x},\nu,t)$$
$$- c\kappa_{abs}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)\ell(\hat{n},\mathbf{x},\nu,t)$$
$$+ c\rho(\mathbf{x},t)\int d^2\hat{n}'[-\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu,t)\ell(\hat{n},\mathbf{x},\nu,t)$$
$$+ \kappa_s(\hat{n}'\to\hat{n};\mathbf{x},\nu,t)\ell(\hat{n}',\mathbf{x},\nu,t)]$$
$$+ j(\mathbf{x},\nu,t)\rho(\mathbf{x},t)/4\pi \qquad (42)$$

We can now look at a situation of radiative equilibrium simplifying our model requiring that the stellar material is not changing (like it would in a general fusion process) and the luminosity function (photon distribution) ℓ remains unchanged. And we would obtain

$$- c\hat{n}\cdot\nabla\ell(\hat{n},\mathbf{x},\nu,t) - c\kappa_{abs}(\mathbf{x},\nu,t)\rho(\mathbf{x},t)\ell(\hat{n},\mathbf{x},\nu,t)$$
$$+ c\rho(\mathbf{x},t)\int d^2\hat{n}'[-\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu,t)\ell(\hat{n},\mathbf{x},\nu,t) + \kappa_s(\hat{n}'\to\hat{n};\mathbf{x},\nu,t)\ell(\hat{n}',\mathbf{x},\nu,t)]$$
$$+ j(\mathbf{x},\nu,t)\rho(\mathbf{x},t)/4\pi = 0 \qquad (43)$$

and of course we assume that κ, j, ρ and ℓ are all independent of time.

We will look now at three fundamental quantities,

1) the radiation energy per volume and per frequency interval

$$\mathcal{E}_{rad}(\mathbf{x},\nu) \equiv \int d^2\hat{n}\ell(\hat{n},\mathbf{x},\nu), \qquad (44)$$

2) the flux vector of radiation energy per frequency interval

$$\Phi_i(\mathbf{x},\nu) \equiv c\int d^2\hat{n}\hat{n}_i\ell(\hat{n},\mathbf{x},\nu), \qquad (45)$$

3) and the spatial part of the energy-momentum tensor of radiation per frequency interval,

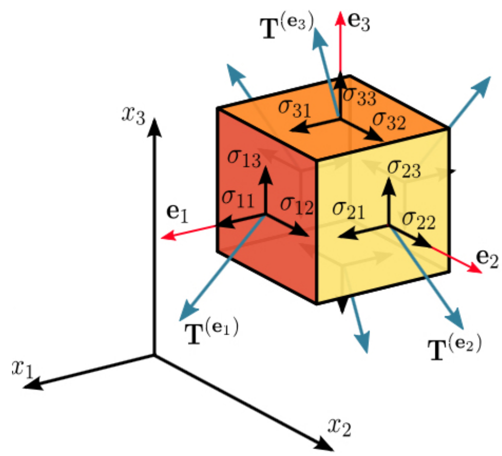

Figure 1: Components of the stress tensor in 3-D. (credit Wikipedia)

$$\Theta_{ij}(\mathbf{x},\nu) \equiv c\int d^2\hat{n}\hat{n}_i\hat{n}_j\ell(\hat{n},\mathbf{x},\nu), \tag{46}$$

Here i and j etc. run over the Cartesian coordinate indices $1,2,3$. Note that $\Phi_i N_i dA d\nu$ is the rate at which radiant energy of frequency between ν and $\nu + d\nu$ passes through a small patch with area dA and unit normal N_i.

The following is a recap of the meaning of the energy momentum tensor.

Energy momentum tensor

The stress–energy tensor, sometimes called the stress–energy–momentum tensor or the energy–momentum tensor, is a tensor physical quantity that describes the density and flux of energy and momentum in spacetime, generalizing the stress tensor of Newtonian physics. It is an attribute of matter, radiation, and non-gravitational force fields. This density and flux of energy and momentum are the sources of the gravitational field in the Einstein field equations of general relativity, just as mass density is the source of such a field in Newtonian gravity.

Stress tensor

In continuum mechanics, the Cauchy stress tensor $\boldsymbol{\sigma}$, true stress tensor, or simply called the stress tensor is a second order tensor named after Augustin-Louis Cauchy. The tensor consists of nine components σ_{ij} that completely define the state of stress at a point inside a material in the deformed state, placement, or configuration. The tensor relates a unit-length direction vector $\hat{n}$ to the traction vector $T(\hat{n})$ across an imaginary surface perpendicular to $\hat{n}$ (the traction vector in the direction $\hat{n}$ is the force in that direction divided by the area under consideration: i.e.a pressure in that direction, see figure 1):

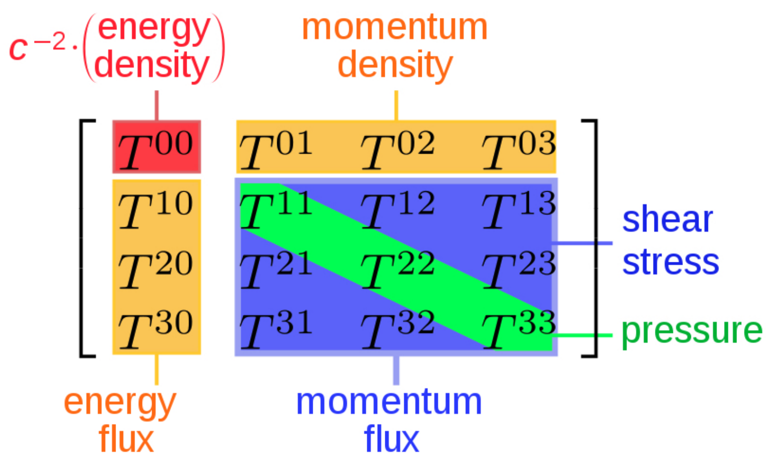

Figure 2: Components of the energy momentum tensor in 4-D. (credit Wikipedia)

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i,$$

or in matrix notation,

$$\begin{bmatrix} T_1^{(\mathbf{n})} & T_2^{(\mathbf{n})} & T_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

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Examples of Energy momentum tensors

I The electromagnetic field strength tensor can be written:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}$$

Maxwell's equations

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J} \tag{47}$$
$$\nabla \cdot \mathbf{E} = \rho \tag{48}$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{49}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{50}$$

In component notation:

$$\epsilon^{ijk}\partial_j B_k - \partial_0 E^i = J^i \tag{51}$$
$$\partial_i E^i = J^0 \tag{52}$$
$$\epsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \tag{53}$$
$$\partial_i B^i = 0 \tag{54}$$

Notice that J^0 is the charge density. So $J^\mu = (\rho, J^x, J^y, J^z)$. Also noticing that $F^{0i} = E^i$ and $F^{ij} = \epsilon^{ijk}B_k$ But this means that we can write (51) and (52) as:

$$\partial_j F^{ij} - \partial_0 F^{0i} = J^i \tag{55}$$
$$\partial_i F^{0i} = J^0 \tag{56}$$

and then combined:

$$\partial_\mu F^{\nu\mu} = J^\nu \tag{57}$$

And (53) and (54):

$$\partial_{[\mu}F_{\nu\lambda]} = 0 \tag{58}$$

II. Describing the momentum and energy in a fluid

A single momentum 4-vector (a vector in 4-D) is not enough to describe the energy and momentum in a fluid. We will define a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ symmetric tensor, $T^{\mu\nu}$, called the energy-momentum tensor or stress-energy tensor. It provides all information we need such as pressure, stress, mass and energy density.

The general definition is the following: "it is the flux of four momentum p^μ across a surface of constant x^ν".i.e. T^{00} is the flux of p^0 across a surface of constant x^0, i.e. the energy density in the rest frame of the fluid. and in identical way $T^{0i} = T^{i0}$ is the momentum density in each spatial direction. T^{ij} is the stress. See figure fig:tensor1.

Off diagonal elements represent shearing terms, like the ones due to viscosity. A diagonal term like T^{11} gives the x-component of the force being exerted per unit of area by a fluid element in the x-direction: this is the x component of the pressure p_x. In general it has three components (no sum here):

$$p^i = T^{ii} \tag{59}$$

Dust

Non interacting matter (like stars far from each other or galaxies, or a very low pressure gas) can be described as dust (one of the quintessential objects loved by physicists). The four-velocity field $U^\mu(x)$ is clearly going to be the constant four-velocity of each particle. If we define the **number-flux-four-vector** to be,

$$N^\mu = nU^\mu, \tag{60}$$

where n is the number density of particles in the rest frame. N^0 is the number of particles measured in any frame (!), while N^i is the flux of particles in the x^i direction. If each particle has the same mass, in the rest frame the energy density of the dust is given by:

$$\rho = mn, \tag{61}$$

Noticing that $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$ we define the energy momentum tensor of dust:

$$T^{\mu\nu}_{dust} = p^\mu N^\nu = mnU^\mu U^\nu = \rho U^\mu U^\nu, \tag{62}$$

In the rest frame of the fluid it is:

$$T^{\mu\nu}_{dust} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{63}$$

A general perfect fluid

A more general form will be:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}, \tag{64}$$

A non zero pressure perfect fluid, can be written in a more general form, in the rest frame of the fluid:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}, \tag{65}$$

where a general perfect fluid is defined by a equation of state $p = p(\rho)$. Examples are $p = \rho$, a stiff fluid, $p = 1/3\rho$ a photon gas, or the cosmological constant $p = -\rho$.

There is a nice compact way to express conservation of momentum and energy with the energy momentum tensor.

$$\partial_\mu T^{\mu\nu} = T^{\mu\nu}{}_{,\mu} = 0, \tag{66}$$

For a perfect fluid:

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho + p)U^\mu U^\nu +$$
$$(\rho + p)(U^\nu \partial_\mu U^\mu + U^\mu \partial_\mu U^\nu) + \partial^\nu p. \tag{67}$$

(Note: $\partial^\nu p = \frac{\partial p}{\partial x^\mu}\eta^\mu{}_\nu = p^{,\nu}$).

We can use this expression (which we will set to zero) further if we look at its projections along the fluid (i.e. the direction of the four-velocity) and in a direction orthogonal to the fluid.