Introduction to Astrophysics
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0.1 Lecture 1
Stars

Emission (thermal and nuclear)

Continuation

Back to the the study of thermal and nuclear emission we can use formulas (44), (45) and (46) to integrate equation (43) in the $\hat{n}$ direction. We obtain

$$\nabla \cdot \Phi(\mathbf{x}, \nu) = -c\kappa_{abs}(\mathbf{x}, \nu)\rho(\mathbf{x})\mathcal{E}_{rad}(\mathbf{x}, \nu) + j(\mathbf{x}, \nu)\rho(\mathbf{x}) \qquad (68)$$

The scattering term in (43) does not contribute because the integrand is antisymmetric in $\hat{n}$ and $\hat{n}'$.

Note: $\kappa_{(}\mathbf{x}, \nu)$, $j(\mathbf{x}, \nu)$, and $\mathcal{E}_{rad}(\mathbf{x}, \nu)$depend only on ν and ρ, the temperature $T(\mathbf{x})$, and chemical composition at $\mathbf{x}$. They may vary with $\mathbf{x}$ as ρ, and $T(\mathbf{x})$ and even the chemical composition vary with $\mathbf{x}$. But if $j(\mathbf{x}, \nu)$ is independent from nuclear processes the medium can come into thermal equilibrium like in a black body cavity, absorbing the thermal emission at each frequency and at each point. We are modeling the real star with a homogeneous medium that at every point has the same temperature, density, and chemical composition that the real one has. For this medium equation (68) requires that $j(\mathbf{x}, \nu) = c\kappa_{abs}\mathcal{E}_{rad}$ With this modeling the star has an emission coefficient given by:

$$j(\mathbf{x}, \nu) = c\kappa_{abs}(\mathbf{x}, \nu)\mathcal{E}_{rad}(\mathbf{x}, \nu) + \epsilon(\mathbf{x}, \nu) \qquad (69)$$

where $\epsilon(\mathbf{x}, \nu)$ is the rate per unit of mass and per frequency interval of energy generated through nuclear reactions. With this definition equation (68) becomes

$$\nabla \cdot \Phi(\mathbf{x}, \nu) = \epsilon(\mathbf{x}, \nu)\rho(\mathbf{x}). \qquad (70)$$

And now we can integrate (43) multiplying first by n_i and then integrate the product over the directions of $\hat{n}$.

$$\nabla_j \Theta_{ij}(\mathbf{x},\nu) = -\kappa_{abs}(\mathbf{x},\nu)\rho(\mathbf{x})\Phi_i(\mathbf{x},\nu)$$
$$-c\rho(\mathbf{x})\int d^2\hat{n}'\int d^2\hat{n}\hat{n}_i[\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu)\ell(\hat{n},\mathbf{x},\nu)$$
$$-\kappa_s(\hat{n}'\to\hat{n};\mathbf{x},\nu)\ell(\hat{n}',\mathbf{x},\nu)] \tag{71}$$

Notice: repeated indeces mean summation. i.e.

$$\nabla_j \Theta_{ij} = \sum_i \frac{\partial}{\partial j}\Theta_{ij} = \frac{\partial}{\partial x_1}\Theta_{i1} + \frac{\partial}{\partial x_2}\Theta_{i2} + \frac{\partial}{\partial x_3}\Theta_{i3}$$

The emission term in (43) ($j(\mathbf{x},\nu,t)\rho(\mathbf{x},t)/4\pi$) does not contribute here, because (j and ρ are independent of photon direction.

If we assume that κ_S is invariant under rotations of the initial and final photon directions, we could define

$$\int d^2\hat{n}'\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu) \equiv \kappa_{out}(\mathbf{x},\nu) \tag{72}$$

and

$$\int d^2\hat{n}\hat{n}_i\kappa_s(\hat{n}'\to\hat{n};\mathbf{x},\nu) \equiv \hat{n}'_i\kappa_{in}(\mathbf{x},\nu) \tag{73}$$

From where it follows that

$$c\int d^2\hat{n}'\int d^2\hat{n}\hat{n}_i\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu)\ell(\hat{n},\mathbf{x},\nu) = \kappa_{out}(\mathbf{x},\nu)\Phi_i(\mathbf{x},\nu) \tag{74}$$

and,

$$c\int d^2\hat{n}'\int d^2\hat{n}\hat{n}_i\kappa_s(\hat{n}'\to\hat{n};\mathbf{x},\nu)\ell(\hat{n}',\mathbf{x},\nu) = \kappa_{in}(\mathbf{x},\nu)\Phi_i(\mathbf{x},\nu) \tag{75}$$

Accordingly

$$c\nabla_j \Theta_{ij}(\mathbf{x},\nu) = -\kappa(\mathbf{x},\nu)\rho(\mathbf{x})\Phi_i(\mathbf{x},\nu), \tag{76}$$

where $\kappa(\mathbf{x},\nu)$ is now the total opacity:

$$\kappa(\mathbf{x},\nu) = \kappa_{abs}(\mathbf{x},\nu) + \kappa_{out}(\mathbf{x},\nu) - \kappa_{in}(\mathbf{x},\nu) \tag{77}$$

To see more clearly the relationship between κ_{in} and κ_{out} we contract equation (73) with $\hat{n}'$ to get

$$\begin{aligned}\hat{n}'_i \cdot \hat{n}'\kappa_{in}(\mathbf{x},\nu) = \kappa_{in}(\mathbf{x},\nu) &= \int d^2\hat{n}(\hat{n}\cdot\hat{n}')\kappa_s(\hat{n}'\to\hat{n};\mathbf{x},\nu) \\ &= \int d^2\hat{n}(\hat{n}\cdot\hat{n}')\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu),\end{aligned} \tag{78}$$

which differs from equation (72) definition of κ_{out} by the factor $\hat{n}\cdot\hat{n}'$. According to Weinberg in his book, textbook treatments of opacity often do not distinguish between absorption and scattering, and so do not encounter the term κ_{in}. This is wrong, because κ_{out} would not vanish even if the scattering were restricted to an infinitesimal neighborhood of the forward direction in which case the scattering should have no effect. The inclusion of κ_{in} removes this paradox, because

$$\kappa_{out}(\mathbf{x},\nu) - \kappa_{in}(\mathbf{x},\nu) = \int d^2\hat{n}'[1-\hat{n}\cdot\hat{n}']\kappa_s(\hat{n}\to\hat{n}';\mathbf{x},\nu) \tag{79}$$

This term vanishes for purely forward scattering, as it must. Weinberg's treatment is not necessary if only Thomson scattering is considered (in that case κ_{in} is zero). But κ_{in} might matter in other scattering, such as bound–bound transitions in which the excited state decays radiatively, with the final photon direction correlated with that of the incoming photon.

The only approximation so far is the one made in (40) in which we assume a short mean free path in the case of radiative energy transport. We will now extend the approximation of short mean free path to the rest of the analysis. We will assume again that the opacity κ is so large that the mean path $1/\kappa\rho$ of typical photons is much smaller than the distance over which conditions vary. This is good enough for the interiors of most stars, but not necessarily true for their outer layers. It follows then that to a good approximation $\ell(\hat{n}, \mathbf{x}, \nu)$ is independent of the photon direction so that Θ_{ij} is approximately proportional to δ_{ij}. From the trace of Eq. (46) we have then

$$\Theta_{ij}(\mathbf{x}, \nu) \simeq \frac{1}{3}\delta_{ij}\mathcal{E}_{rad}(\mathbf{x}, \nu). \tag{80}$$

With $1/\kappa\rho$ very short the radiation is in thermal equilibrium with local matter at a temperature T, so that

$$\mathcal{E}_{rad}(\mathbf{x}, \nu) \simeq B(\nu, T(\mathbf{x}), \tag{81}$$

where B is the Planck black-body distribution

$$B(\nu, T(\mathbf{x})) = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp(h\nu/k_B T) - 1} \tag{82}$$

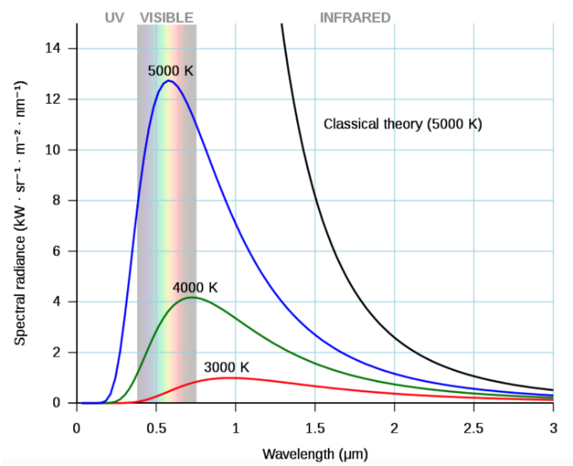

Figure 1: A family of Planck curves for different temperatures. The classical (black) curve diverges from observed intensity at high frequencies (short wavelengths). (credit Wikipedia)

We use equations (80) and (81) and plug them into eq (76) to obtain

$$c\nabla B(\nu, T(\mathbf{x}) = -3\kappa(\mathbf{x},\nu)\rho(\mathbf{x})\Phi(\mathbf{x},\nu), \tag{83}$$

$\ell(\hat{n}, \mathbf{x}, \nu)$ does depend on $\hat{n}$ (up and down inside the star is not the same). But we are neglecting it in equation (80) and (81) but since $\kappa\rho$ is assumed large ($1/\kappa\rho$ small), we may not neglect the quantity $\kappa\rho\Phi_i$ in Eq. (83), even though perfect isotropy of the photon distribution would make Φ_i vanish. So we can now study the simple case of spherical symmetry in which the only special direction at any point is the radial direction, which distinguishes up and down. We then take the flux vector to point in the direction $\hat{x} \equiv \mathbf{x}/r$ and depend only on ν and $r = |x|$, so that we may write

$$\Phi(\mathbf{x}, \nu) = \hat{x}\frac{\mathcal{L}(r,\nu)}{4\pi r^2}. \tag{84}$$

where now $\mathcal{L}(r,\nu)$ is the total energy radiation flux per time, per frequency interval outward through a sphere of radius r.

In this case then eqs (70) and (83) give the following differential equation for $\mathcal{L}(r,\nu)$

$$\frac{d\mathcal{L}(r,\nu)}{dr} = 4\pi r^2 \epsilon(r,\nu)\rho(r), \tag{85}$$

where as defined before $\epsilon(\mathbf{x},\nu)$ is the rate per unit of mass and per frequency interval of energy generated through nuclear reactions. And in consequence,

$$c\frac{dB(\nu,T(r))}{dr} = -3\kappa(r,\nu)\rho(r)\frac{\mathcal{L}(r,\nu)}{4\pi r^2} \tag{86}$$