Introduction to Astrophysics
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The following is a refresher of topics usually covered in Introductory Astronomy and Modern Physics.

BLACKBODY RADIATION
(The following notes are based on material from Carroll & Ostlie, An Introduction to Modern Astrophysics, 2nd edition)

The Orion belt is shown in Fig. 1

Figure 1:

Betelgeuse on the top left has a surface temperature of roughly 3600 K, significantly cooler than the 13,000-K surface of Rigel on the bottom right.

Connection between Color and Temperature

Very early observations on the properties of thermal radiation were documented in 1792 by Thomas Wedgewood, the famous British manufacturer of Wedgewood porcelain. Wedgewood observed that all objects in his ovens, where he prepared fine china, would glow the same red color when

they reached a certain temperature, regardless of their size, shape, or composition. It was not until 1859 when Gustav Kirchhoff demonstrated that, for any body in thermal equilibrium with radiation, the emitted energy is proportional to the energy absorbed by the object. An example of such a case would be the heated walls of a clay oven (kiln) with its door closed and at a constant temperature. The radiation within the walls of the kiln would be in thermal equilibrium when the radiation energy within the kiln is absorbed, exchanged, and reemitted many times over until the entire walls of the cavity of the kiln are in thermal equilibrium. The radiation in thermal equilibrium within the walls of a kiln is similar to the radiation emitted by a black body, which is an object that absorbs radiation of all wavelengths or frequencies and therefore would appear black. A black body emits the energy it absorbs in accord with Kirchhoff's observations; and the energy emitted is a function of the temperature of the black body and frequency of the emitted light, and independent of the size, shape, and chemical nature of the black body.
William Wien, in 1893, mathematically defined the spectral density of a black-body cavity, that is, the energy per unit volume per unit frequency within a black-body cavity, as a function of black-body temperature. The equation derived by Wien became known as Wien's exponential law, because the energy density was an exponential function of the radiation frequency and black-body temperature. Radiation spectroscopists at the time determined experimentally that Wien's law fit well for the short wavelengths of radiation (0–4 μm) over a wide range of temperatures (400–1600 K), but that the law failed for longer radiation wavelengths. This Wien's law:

$$\lambda_{max} T = 0.002897755 \text{ mK}. \quad (83)$$

Figure 2 shows that as the temperature of a blackbody increases, it emits more energy per second at all wavelengths. Experiments performed by

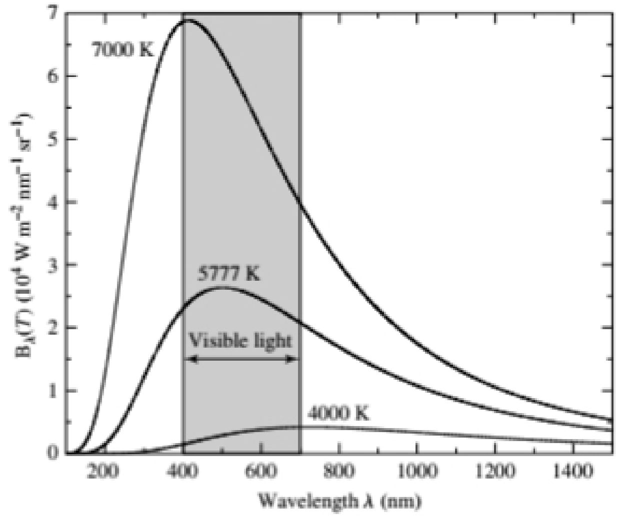

FIGURE 3.8 Blackbody spectrum [Planck function $B_\lambda(T)$].

Figure 2: A family of Planck curves for different temperatures. Credit: Carroll &Ostlie.

Josef Stefan in 1879 showed that the luminosity,$\mathcal{L}$, of a blackbody of area A and temperature T (in the absolute temperature scale) is given by

$$\mathcal{L} = A\sigma T^4 \tag{84}$$

Five years later Ludwig Boltzmann (1844–1906), derived this equation, now called the Stefan–Boltzmann equation, using the laws of thermodynamics and Maxwell's formula for radiation pressure. The Stefan–Boltzmann constant, σ, has the value $\sigma = 5.67040010^8\ \mathrm{Wm^{-2}K^{-4}}$. For a spherical star of radius R and surface area $A = 4\pi R^2$, the Stefan–Boltzmann equation takes the form

$$\mathcal{L} = 4\pi R^2 \sigma T^4 \tag{85}$$

Since stars are not perfect blackbodies, we use this equation to define the effective temperature T_{eff} of a star's surface. Combining this with the inverse square law, shows that at the surface of the star (r = R), the surface flux is $\Phi_{surf} = \sigma T_e^4$.

Quantization of energy

One of the problems haunting physicists at the end of the nineteenth century was their inability to derive from fundamental physical principles the blackbody radiation plot from figure 2. John William Strutt (Lord Rayleigh) attempted to arrive at the expression by applying Maxwell's equations of classical electromagnetic theory together with the results from thermodynamics.

He treated black body radiation as stationary radiation (standing waves) inside an oven. If L is the distance between the oven's walls, then the permitted wavelengths of the radiation are $\lambda = 2L, L, 2L/3, 2L/4, 2L/5, ...$ extending to infinitely increasing shorter wavelengths for. According to classical physics, each of these wavelengths should receive an amount of energy equal to kT, where $k = 1.3806503 \times 10^{-23}$ J/ K is Boltzmann's constant, from the ideal gas law $PV = NkT$. The result of Rayleigh's derivation gave was

$$B_\lambda(T) = \frac{2ckT}{\lambda^4}, \tag{86}$$

(valid only if λ is long) which agrees well with the long-wavelength tail of the blackbody radiation curve. However, a severe problem with Rayleigh's result is that this solution for $B_\lambda(T) \to \infty$ when $\lambda(T) \to 0$.

The source of the problem is that according to classical physics, an infinite number of infinitesimally short wavelengths implied that an unlimited amount of blackbody radiation energy was contained in the oven, an absurd theoretical result that was dubbed the "ultraviolet catastrophe". Equation (86) is known today as the Rayleigh–Jeans law.

At the same time Wien was also working on developing the correct mathematical expression for the blackbody radiation curve. Guided by the Stefan-Boltzmann law (Eq. 85) and classical thermal physics, Wien was able to develop an empirical law that described the curve at short wavelengths but failed at longer wavelengths:

$$B_\lambda(T) \simeq a\lambda^{-5}e^{-b/\lambda T}, \tag{87}$$

(valid only if λ is short) where a and b were arbitrary constants chosen to provide the best fit to the experimental data.

ln 1900 the German physicist Max Planck had discovered that a modification of Wien's expression could be made to fit the blackbody spectra shown in figure 2 while simultaneously replicating the long-wavelength success of the Rayleigh–Jeans law and avoiding the ultraviolet catastrophe:

$$B_\lambda(T) = \frac{a/\lambda^5}{e^{b/\lambda T} - 1}, \tag{88}$$

In order to determine the constants a and b, Planck assumed that a standing electromagnetic wave of wavelength λ and frequency $\nu = c/\lambda$ could not acquire just any arbitrary amount of energy. Instead, the wave could have only specific allowed energy values that were integral multiples of a minimum wave energy. This minimum energy, a quantum of energy, is given by $h\nu$ or hc/λ, where h is now called the Planck constant. Thus the energy of an electromagnetic wave is $nh\nu$ or nhc/λ, where n (an integer) number is the number of quanta in the wave. Given this assumption of quantized wave energy with a minimum energy proportional to the frequency of the wave, the entire oven could not contain enough energy to supply even one quantum of energy for the short-wavelength, high-frequency waves. Thus the ultraviolet catastrophe would be avoided. His formula, now known as the Planck function, agreed wonderfully with experiment, but only if the constant h remained in the equation with the value $h = 6.62606876 \times 10^{-34}$ J s.

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}, \tag{89}$$

we can now apply Planck's function to astrophysical systems. In spherical coordinates, the amount of radiant energy per unit time having wavelengths betweenλ and $\lambda + d\lambda$ emitted by a blackbody of temperature T and surface area dA into a solid angle $d\Omega = \sin\theta d\theta d\phi$ is given by

$$B_\lambda(T) d\lambda dA \cos\theta d\Omega = B_\lambda(T) d\lambda dA \cos\theta \sin\theta d\theta d\phi; \tag{90}$$

The units of $B_\lambda(T)$ are therefore $\mathrm{Wm}^{-3}\mathrm{s}r^{-1}$. Unfortunately, these units can be misleading. We note that Wm^{-3} indicates power (energy per unit time) per unit area per unit wavelength interval, not energy per unit time per unit volume. To help avoid confusion, the units of the wavelength interval $d\lambda$ are sometimes expressed in nanometers rather than meters, so the units of the Planck function become $\mathrm{Wm}^{-2}\mathrm{nm}^{-1}\mathrm{sr}^{-1}$ as in Fig. 2.

Many times it is more convenient to deal with frequency intervals $d\nu$ than with wavelength intervals $d\lambda$. Remember that a photon energy is precisely $h\nu$ which makes for more appropriate units for expressing a distribution of energies as function of temperature. In this case the Planck function has the form

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}, \tag{91}$$

The Planck function can be used to make the connection between the observed properties of a star (radiant flux, apparent magnitude) and its intrinsic properties (radius, temperature). Consider an ideal star consisting of a spherical blackbody of radius R and temperature T. Assuming that each small patch of surface area dA emits blackbody radiation isotropically (equally in all directions) over the outward hemisphere, the energy per second having wavelengths between λ and $\lambda + d\lambda$ emitted by the star is

$$\mathcal{L}_\lambda d\lambda = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} B_\lambda d\lambda dA \cos\theta \sin\theta d\theta d\phi; \tag{92}$$

The angular integration yields a factor of π, and the integral over the area of the sphere produces a factor of $4\pi R^2$. The result is

$$\begin{aligned}\mathcal{L}_\lambda d\lambda &= 4\pi^2 R^2 B_\lambda d\lambda \\ &= \frac{8\pi^2 R^2 hc^2/\lambda^5}{e^{hc/\lambda kT}-1} d\lambda \end{aligned} \tag{93}$$

$L_\lambda d\lambda$ is known as the monochromatic luminosity. Comparing the Stefan–Boltzmann equation (3) with the result of integrating Eq. (11) over all wavelengths we get:

$$\int_0^\infty B_\lambda d\lambda = \frac{\sigma T^4}{\pi}; \tag{94}$$

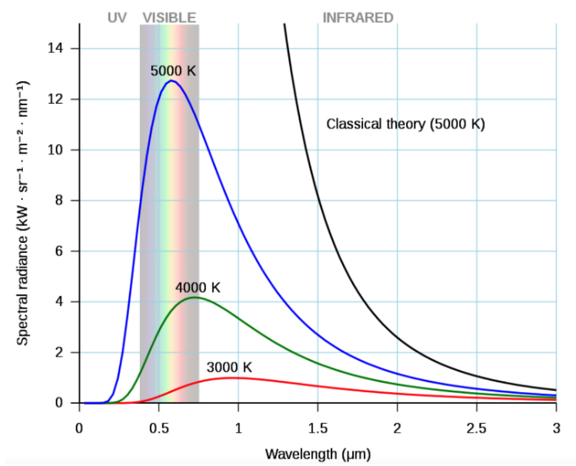

Figure 3: A family of Planck curves for different temperatures. The classical (black) curve diverges from observed intensity at high frequencies (short wavelengths). (credit Wikipedia)

We can now go back to the energy-momentum tensor of radiation from formula (80-81) and use eq (76) to obtain for $B(\nu, T(\mathbf{x})$ the following equation:

$$c\nabla B(\nu, T(\mathbf{x}) = -3\kappa(\mathbf{x},\nu)\rho(\mathbf{x})\Phi(\mathbf{x},\nu), \tag{95}$$

Although $\ell(\hat{n}, \mathbf{x}, \nu)$ does depend on $\hat{n}$ (up and down inside the star is not the same) we are neglecting it in equation (80) and (81). But since $\kappa\rho$ is assumed large ($1/\kappa\rho$ small), we may not neglect the quantity $\kappa\rho\Phi_i$ in Eq. (95), even though perfect isotropy of the photon distribution would make Φ_i vanish. So we can now study the simple case of spherical symmetry in which the only special direction at any point is the radial direction, which distinguishes up and down. We then take the flux vector to point in the direction $\hat{x} \equiv \mathbf{x}/r$ and depend only on ν and $r \equiv |\mathbf{x}|$, so that we may write

$$\Phi(\mathbf{x}, \nu) = \hat{x}\frac{\mathcal{L}(r, \nu)}{4\pi r^2}. \tag{96}$$

where now $\mathcal{L}(r, \nu)$ is the total energy radiation flux per time, per frequency interval outward through a sphere of radius r.

In this case then eqs (70) and (95) give the following differential equation for $\mathcal{L}(r, \nu)$

$$\frac{d\mathcal{L}(r, \nu)}{dr} = 4\pi r^2 \epsilon(r, \nu)\rho(r), \tag{97}$$

where as defined before $\epsilon(\mathbf{x}, \nu)$ is the rate per unit of mass and per frequency interval of energy generated through nuclear reactions. And in consequence,

$$c\frac{dB(\nu, T(r))}{dr} = -3\kappa(r, \nu)\rho(r)\frac{\mathcal{L}(r, \nu)}{4\pi r^2} \tag{98}$$

To calculate now the temperature distribution in a star all we need to do is to integrate the total radiative energy for all frequencies,

$$\mathcal{L}(r) \equiv \int \mathcal{L}(r, \nu) d\nu, \tag{99}$$

and the total energy per gram emitted by nuclear processes at all frequencies is

$$\epsilon(r) \equiv \int \epsilon(r, \nu) d\nu. \tag{100}$$

$$\frac{d\mathcal{L}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r). \tag{101}$$

To write an equation for dT/dr in terms of $\mathcal{L}(r)$ we can use eq (98) dividing it by $\kappa(r,\nu)$ and integrate over the range of frequencies,

$$-3\rho(r)\frac{\mathcal{L}(r,\nu)}{4\pi r^2} = c\int d\nu \frac{1}{\kappa(r,\nu)}\left(\frac{\partial B(\nu,T(r)}{\partial T}\right)_{T=T(r)} T'(r). \tag{102}$$

Opacity is clearly a function of the medium and the radiation frequency. A crucial indicator of opacity in stars is the **Rosseland mean opacity**. To define it we continue working with eq (102) and define a κ that it does not depend on the frequency – in some sense it is an average –.

$$\kappa(r) \equiv \frac{\int d\nu \left(\frac{\partial B(\nu,T}{\partial T}\right)_{T=T(r)}}{\int d\nu \frac{1}{\kappa(r,\nu)}\left(\frac{\partial B(\nu,T(r)}{\partial T}\right)_{T=T(r)}} \tag{103}$$

From Planck distribution to Stefan Boltzman

Planck's distribution is given by eq (82)

$$B(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{\exp(h\nu/k_B T) - 1}$$

Integrating over all frequencies

$$\int d\nu \left(\frac{\partial B(\nu, T}{\partial T} \right)_{T=T(r)} = 4aT^3(r) \tag{104}$$

where a is the Stefan-Boltzman constant (radiation energy constant as empirically found by Stefan in his law providing the energy density in a black body as a function of its temperature: $u = \frac{U}{V} = aT^4$. Its value in terms of the Boltzman and Planck's constant is

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3} = 7.566 \times 10^{-15} erg\ cm^{-3} K^{-4} \tag{105}$$

We can now write eq (102) as

$$-3\rho(r)\frac{\mathcal{L}(r)}{4\pi r^2} = \frac{4acT^3(r)T'(r)}{\kappa(r)} \tag{106}$$

or solving for the temperature gradient,

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)\mathcal{L}(r)}{4acT^3(r)4\pi r^2} \tag{107}$$

Eq (101) and (107) are the **fundamental equations of radiative energy transport** in spherical star interiors.

An opacity function and its Rosseland mean can be redefined to make them dependent on density and temperature rather than on position, i.e. a "$\kappa(\rho, T, \nu)$, and its mean $\kappa(\rho, T)$:

$$\kappa(r,\nu) = \kappa(\rho(r), T(r), \nu), \qquad \kappa(r) = \kappa(\rho(r), T(r)) \tag{108}$$

with it the Rosseland mean takes the position-independent form

$$\int d\nu \frac{1}{\kappa(\rho, T, \nu)} \left(\frac{\partial B(\nu, T)}{\partial T} \right) = \frac{4aT^3}{\kappa(\rho, T)} \tag{109}$$