

Introduction to Astrophysics
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Free-Free Absorption

In the absence of external fields, the conservation of energy and momentum forbids the absorption of a photon by a free electron. If the photon has momentum \vec{q} then it has energy $c|\vec{q}|$, so the conservation of energy and momentum requires that

$$0 = (E' - E)^2 - c^2(\vec{p}' - \vec{p})^2 = 2m_e^2c^4 - 2E'E + 2c^2\vec{p}' \cdot \vec{p}. \quad (136)$$

where \vec{p}' and \vec{p} are the initial and final electron momenta, and $E = [c^2\vec{p}^2 + m_e^2c^4]^{1/2}$ and $E' = [c^2\vec{p}'^2 + m_e^2c^4]^{1/2}$ are the initial and final electron energies.

Conservation of energy requires that in the rest frame of the electron where it is originally $E = m_e c^2$ it remains $E' = m_e c^2$. The story is different in the Coulomb field of an atomic nucleus where *bremsstrahlung* occurs. The absorption of photons by free electrons in the Coulomb field of a nucleus leads to what is known as Kramers' opacity. Kramers' was a classical result showing that in ef-



Figure 1: Bremsstrahlung

fect the rate of absorption of a photon of frequency ν (averaged over photon directions and helicities) is

$$\Gamma_{Kramers}(\rho, T, \nu) = \int n_e(\mathbf{v}, T) d^3v \frac{4\pi Z^2 e^6 n_N}{3\sqrt{3} h m_e^2 \nu^3} \quad (137)$$

The photon also carries a quantity called spin angular momentum that does not depend on its frequency. Because photons always move at the speed of light, the spin is best expressed in terms of the component measured along its direction of motion, its helicity, which must be either $+\hbar$ or $-\hbar$. These two possible helicities, called right-handed and left-handed, correspond to the two possible circular polarization states of the photon. The formula above is a classical one because it does not include a quantum treatment of radiation (thus the lack of the Planck constant).

The integral in (137) is over initial electron velocities \mathbf{v} ; $n_e(\nu, T)$ is the number of electrons per spatial volume and per velocity-space volume; n_N is the number density of ions, taken to have charge Ze ; e is the magnitude of the electron charge in Gaussian electrostatic units; and $h = 2\pi\hbar$. This can be significantly modified by quantum and other corrections. But with or without these corrections, the net rate $c\rho\kappa$ of photon absorption in free-free transitions is quadratic in particle densities, so $\alpha = 1$, but the temperature dependence is more complicated. It was first calculated by Gaunt. It has become traditional to express the rate per electron as the Kramers result multiplied by a correction factor, known as the free-free Gaunt factor:

$$\Gamma_{ff\,abs}(\rho, T, \nu) = \int n_e(\mathbf{v}, T) d^3v \frac{4\pi Z^2 e^6 n_N}{3\sqrt{3} h m_e^2 \nu^3} g_{ff}(\nu, v). \quad (138)$$

The calculation of g_{ff} could be quite complicated. It is not so difficult to do the calculation in the so called Born approximation (a perturbation method applied to scattering by an extended body which is accurate if the scattered field is small compared to the incident field on the scatterer). This approximation is first order in the Coulomb potential. In this order the rate at which a photon of frequency ν is absorbed is

$$\Gamma_{ff\,abs}(\rho, T, \nu) = \int n_N n_e(\mathbf{v}, T) d^3v \frac{4Z^2 e^6}{3h m_e^2 \nu^3} \ln \left(\frac{v' + v}{v' - v} \right), \quad (139)$$

where v' is the final electron velocity, given by the energy conservation condition

$$\frac{m_e v'^2}{2} = \frac{m_e v^2}{2} + h\nu \quad (140)$$

which implies that the Gaunt factor is given by

$$g_{ff}(\nu, v) = \frac{\sqrt{3}}{\pi} \ln \left(\frac{v' + v}{v' - v} \right), \quad (141)$$

This is a good approximation for non-relativistic electrons if the Coulomb potential at an electron scattered by an atom or an ion is typically much less than the involved electron kinetic energies, which is the case if $Ze^2/\hbar\nu \ll 1$ and $Ze^2/\hbar v' \ll 1$.

In thermal equilibrium at temperature T , far from degeneracy, the electron velocity distribution is given by the Maxwell–Boltzmann formula

$$n_e(\mathbf{v}, T) = n_e \left(\frac{m_e}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{m_e v^2}{2k_B T} \right), \quad (142)$$

where n_e is the total electron number density. We can find the temperature dependance of the equation (139) giving the rate of absorption of a photon of frequency ν defining a rescaled variable $x \equiv v\sqrt{m_e/2k_B T}$. It becomes then

$$\Gamma_{ff\,abs}(\rho, T, \nu) = n_e n_N \frac{16Z^2 e^6}{3hcm_e^2 \nu^3} \int_0^\infty x e^{-x^2} dx \times \ln \left(\frac{x' + x}{x' - x} \right), \quad (143)$$

where n_e and n_N are the total number densities of electrons and ions, respectively. We can now add a correction factor to account for stimulated emission $1 - \exp(-h\nu/k_B T)$ and write the result as $c\rho\kappa_{ff}$,

$$\begin{aligned} \kappa_{ff}(\rho, T, \nu) &= \rho N_e N_N \frac{16Z^2 e^6}{3hcm_e^2 \nu^3} \sqrt{\frac{m_e}{2\pi k_B T}} \int_0^\infty x e^{-x^2} dx \\ &\times \ln \left(\frac{x' + x}{x' - x} \right) (1 - \exp(-h\nu/k_B T)), \end{aligned} \quad (144)$$

where $N_e = n_e/\rho$ is the number of electrons per gram, $N_N = n_N/\rho$ is the number of nuclei per gram, and is given by the energy conservation equation (140)

$$\frac{m_e v'^2}{2} = \frac{m_e v^2}{2} + h\nu \quad (140)$$

as

$$x'^2 = x^2 + y, \quad y \equiv h\nu/k_B T. \quad (145)$$

The Rosseland mean opacity defined in

$$\int d\nu \frac{1}{\kappa(\rho, T, \nu)} \left(\frac{\partial B(\nu, T)}{\partial T} \right) = \frac{4aT^3}{\kappa(\rho, T)}$$

is then

$$\kappa(\rho, T) = \frac{8\rho(k_B T)^{-7/2} N_e N_N Z^2 e^6 h^6 (a/k_B^4) m_e^{-3/2}}{3\sqrt{2}\pi^{3/2} \int_0^\infty dy \frac{y^6 e^y}{(e^y - 1)} \left[\int_0^\infty x e^{-x^2} dx \times \ln \left(\frac{x'+x}{x'-x} \right) \right]^{-1}}, \quad (146)$$

with x' related to the integration variables x and y by the energy conservation condition (145). The important result is that in Eq. (131) the Kramers opacity has

$$\alpha = 1, \quad \beta = -7/2. \quad (147)$$

The mean opacity has a factor $T^{-1/2}$ extra in Eq. (144), adding to the expected $1/T^3$ from the Rosseland mean (converted from $1/\nu^3$ in (144)). The $T^{-7/2}$ dependence of the free-free opacity does not continue to arbitrary low temperatures. At very low temperatures the free-free and Thomson scattering contributions to the opacity both become negligible because there are very few free electrons.