Introduction to Astrophysics
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Homework 3

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BLACKBODY RADIATION (The following notes are based on material from Carroll & Ostlie, An Introduction to Modern Astrophysics, 2nd edition)

The Orion belt is shown in Fig. 2

Figure 1:

Betelgeuse on the top left has a surface temperature of roughly 3600 K, significantly cooler than the 13,000-K surface of Rigel on the bottom right.

Connection between Color and Temperature

Very early observations on the properties of thermal radiation were documented in 1792 by Thomas Wedgewood, the famous British manufacturer of Wedgewood porcelain. Wedgewood observed that all objects in his ovens, where he prepared fine china, would glow the same red color when they reached a certain temperature, regardless of their size, shape, or composition. It was

not until 1859 when Gustav Kirchhoff demonstrated that, for any body in thermal equilibrium with radiation, the emitted energy is proportional to the energy absorbed by the object. An example of such a case would be the heated walls of a clay oven (kiln) with its door closed and at a constant temperature. The radiation within the walls of the kiln would be in thermal equilibrium when the radiation energy within the kiln is absorbed, exchanged, and reemitted many times over until the entire walls of the cavity of the kiln are in thermal equilibrium. The radiation in thermal equilibrium within the walls of a kiln is similar to the radiation emitted by a black body, which is an object that absorbs radiation of all wavelengths or frequencies and therefore would appear black. A black body emits the energy it absorbs in accord with Kirchhoff's observations; and the energy emitted is a function of the temperature of the black body and frequency of the emitted light, and independent of the size, shape, and chemical nature of the black body.
William Wien, in 1893, mathematically defined the spectral density of a black-body cavity, that is, the energy per unit volume per unit frequency within a black-body cavity, as a function of black-body temperature. The equation derived by Wien became known as Wien's exponential law, because the energy density was an exponential function of the radiation frequency and black-body temperature. Radiation spectroscopists at the time determined experimentally that Wien's law fit well for the short wavelengths of radiation (0–4 μm) over a wide range of temperatures (400–1600 K), but that the law failed for longer radiation wavelengths. This Wien's law:

$$\lambda_{max} T = 0.002897755 \text{ mK}. \tag{1}$$

Exercise 1

a) Using Wien's law estimate Betelgeuse's surface temperature knowing that its brightest wavelength is 805 nm.

b) Similarly estimate Rigel's temperature knowing that its brightest wavelength is 223 nm. Elaborate about the colors observed in Figure 1.

Figure 2 shows that as the temperature of a blackbody increases, it emits more energy per second at all wavelengths. Experiments performed by Josef Stefan in 1879 showed that the luminosity,$\mathcal{L}$, of a blackbody of area A and temperature T (in the absolute temperature scale) is given by

$$\mathcal{L} = A\sigma T^4 \tag{2}$$

Five years later Ludwig Boltzmann (1844–1906), derived this equation, now called the Stefan–Boltzmann equation, using the laws of thermodynamics and Maxwell's formula for radiation pressure. The Stefan–Boltzmann constant, σ, has the value $\sigma = 5.670400 10^{8}$ $\text{Wm}^{-2}\text{K}^{-4}$. For a spherical star of radius R and surface area $A = 4\pi R^2$, the Stefan–Boltzmann equation takes the

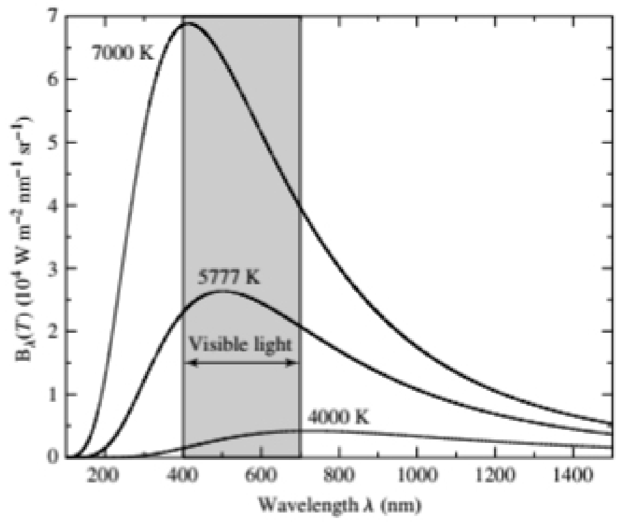

FIGURE 3.8 Blackbody spectrum [Planck function $B_\lambda(T)$].

Figure 2: A family of Planck curves for different temperatures. Credit: Carroll &Ostlie.

form

$$\mathcal{L} = 4\pi R^2 \sigma T^4 \tag{3}$$

Since stars are not perfect blackbodies, we use this equation to define the effective temperature T_{eff} of a star's surface. Combining this with the inverse square law, shows that at the surface of the star (r = R), the surface flux is $\Phi_{surf} = \sigma T_e^4$.

Exercise 2

The luminosity of the Sun is $\mathcal{L}_\odot = 3.839 \times 10^{26} W$ and its radius is $R_\odot = 6.95508 \times 10^8 m$. The effective temperature of the Sun's surface.
a) Calculate the effective temperature of the Sun's surface and
b) its radiant flux at the solar surface.

Quantization of energy

One of the problems haunting physicists at the end of the nineteenth century was their inability to derive from fundamental physical principles the blackbody radiation plot from figure 2. John William Strutt (Lord Rayleigh) attempted to arrive at the expression by applying Maxwell's equations of classical electromagnetic theory together with the results from thermodynamics.

He treated black body radiation as stationary radiation (standing waves) inside an oven. If L is the distance between the oven's walls, then the permitted wavelengths of the radiation are $\lambda = 2L, L, 2L/3, 2L/4, 2L/5, ...$ extending to infinitely increasing shorter wavelengths for. According to classical physics, each of these wavelengths should receive an amount of energy equal to kT, where $k = 1.3806503 \times 10^{-23}$ J/ K is Boltzmann's constant, from the ideal gas law $PV = NkT$. The result of Rayleigh's derivation gave was

$$B_\lambda(T) = \frac{2ckT}{\lambda^4}, \tag{4}$$

(valid only if λ is long) which agrees well with the long-wavelength tail of the blackbody radiation curve. However, a severe problem with Rayleigh's result is that this solution for $B_\lambda(T) \to \infty$ when $\lambda(T) \to 0$.

The source of the problem is that according to classical physics, an infinite number of infinitesimally short wavelengths implied that an unlimited amount of blackbody radiation energy was contained in the oven, an absurd theoretical result that was dubbed the "ultraviolet catastrophe". Equation (4) is known today as the Rayleigh–Jeans law.

At the same time Wien was also working on developing the correct mathematical expression for the blackbody radiation curve. Guided by the Stefan-Boltzmann law (Eq. 3) and classical thermal physics, Wien was able to develop an empirical law that described the curve at short wavelengths but failed at longer wavelengths:

$$B_\lambda(T) \simeq a\lambda^{-5} e^{-b/\lambda T}, \tag{5}$$

(valid only if λ is short) where a and b were arbitrary constants chosen to provide the best fit to the experimental data.

ln 1900 the German physicist Max Planck had discovered that a modification of Wien's expression could be made to fit the blackbody spectra shown in figure 2 while simultaneously replicating the long-wavelength success of the Rayleigh–Jeans law and avoiding the ultraviolet catastrophe:

$$B_\lambda(T) = \frac{a/\lambda^5}{e^{b/\lambda T} - 1}, \tag{6}$$

In order to determine the constants a and b, Planck assumed that a standing electromagnetic wave of wavelength λ and frequency $\nu = c/\lambda$ could not acquire just any arbitrary amount of energy. Instead, the wave could have only specific allowed energy values that were integral multiples of a minimum wave energy. This minimum energy, a quantum of energy, is given by $h\nu$ or hc/λ, where h is now called the Planck constant. Thus the energy of an electromagnetic wave is $nh\nu$ or nhc/λ, where n (an integer) number is the number of quanta in the wave. Given this assumption of quantized wave energy with a minimum energy proportional to the frequency of the wave, the entire oven could not contain enough energy to supply even one quantum of energy for the short-wavelength, high-frequency waves. Thus the ultraviolet catastrophe would be avoided. His formula, now known as the Planck function, agreed wonderfully with experiment, but only if the constant h remained in the equation with the value $h = 6.62606876 \times 10^{-34}$ J s.

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}, \tag{7}$$

we can now apply Planck's function to astrophysical systems. In spherical coordinates, the amount of radiant energy per unit time having wavelengths betweenλ and $\lambda + d\lambda$ emitted by a blackbody of temperature T and surface area dA into a solid angle $d\Omega = \sin\theta d\theta d\phi$ is given by

$$B_\lambda(T) d\lambda dA \cos\theta d\Omega = B_\lambda(T) d\lambda dA \cos\theta \sin\theta d\theta d\phi; \tag{8}$$

The units of $B_\lambda(T)$ are therefore $\mathrm{Wm}^{-3}\mathrm{s}r^{-1}$. Unfortunately, these units can be misleading. We note that Wm^{-3} indicates power (energy per unit time) per unit area per unit wavelength interval, not energy per unit time per unit volume. To help avoid confusion, the units of the wavelength interval $d\lambda$ are sometimes expressed in nanometers rather than meters, so the units of the Planck function become $\mathrm{Wm}^{-2}\mathrm{nm}^{-1}\mathrm{sr}^{-1}$ as in Fig. 2.

Many times it is more convenient to deal with frequency intervals $d\nu$ than with wavelength intervals $d\lambda$. Remeber that a photon energy is precisely $h\nu$ which makes for more appropriate units for expressing a distribution of energies as function of temperature. In this case the Planck function has the form

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}, \tag{9}$$

The Planck function can be used to make the connection between the observed properties of a star (radiant flux, apparent magnitude) and its intrinsic properties (radius, temperature). Consider an ideal star consisting of a spherical blackbody of radius R and temperature T. Assuming that each small patch of surface area dA emits blackbody radiation isotropically (equally in all directions) over the outward hemisphere, the energy per second having wavelengths between λ and $\lambda + d\lambda$ emitted by the star is

$$\mathcal{L}_\lambda d\lambda = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} B_\lambda d\lambda dA \cos\theta \sin\theta d\theta d\phi; \tag{10}$$

The angular integration yields a factor of π, and the integral over the area of the sphere produces a factor of $4\pi R^2$. The result is

$$\begin{aligned} \mathcal{L}_\lambda d\lambda &= 4\pi^2 R^2 B_\lambda d\lambda \\ &= \frac{8\pi^2 R^2 hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} d\lambda \end{aligned} \tag{11}$$

$L_\lambda d\lambda$ is known as the monochromatic luminosity. Comparing the Stefan–Boltzmann equation (3) with the result of integrating Eq. (11) over all wavelengths we get:

$$\int_0^\infty B_\lambda d\lambda = \frac{\sigma T^4}{\pi}; \qquad (12)$$

Exercise 3

(a) Show that the Rayleigh–Jeans law (Eq. 4) is an approximation of the Planck function B_λ in the limit of $\lambda \gg b/T$ from equation (6). (Use the first-order expansion $e^x \approx 1 + x$for $x \ll 1$. Notice that Planck's constant is not present in your answer. The Rayleigh–Jeans law is a classical result, so the "ultraviolet catastrophe" at short wavelengths, produced by the λ^4 in the denominator, cannot be avoided.
(b) Plot the Planck function B_λ and the Rayleigh–Jeans law for the Sun ($T = 5777$ K) on the same graph. At roughly what wavelength is the Rayleigh–Jeans value twice as large as the Planck function?

Exercise 4

(a) Integrate Eq. (11) over all wavelengths to obtain an expression for the total luminosity of a blackbody model star. You will need to use that

$$\int_0^\infty \frac{u^3 du}{e^u - 1} = \frac{\pi^4}{15} \qquad (13)$$

(b) Compare your result with the Stefan–Boltzmann equation (3), and show that the Stefan-Boltzmann constant σ is given by

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}.$$

(c) Calculate the value of σ from this expression, and compare with $\sigma = 5.670400 \times 10^{-8} W m^{-2} K^{-4}$.

The following exercises are only for the graduate students who are taking the PHYS 6381 class.

Exercise 5

Equation numbers referred to the lecture notes from class. Use formulas (44), (45) and (46) to integrate equation (43) in the $\hat{n}$ direction showing that you obtain formula (68)

$$\nabla \cdot \Phi(\mathbf{x}, \nu) = -c\kappa_{abs}(\mathbf{x}, \nu)\rho(\mathbf{x})\mathcal{E}_{rad}(\mathbf{x}, \nu) + j(\mathbf{x}, \nu)\rho(\mathbf{x})$$

Exercise 6

Show that with the definition (69), equation (68) becomes formula (70).

Exercise 7

a) Use equations (80) and (81) and (76) to show that the gradient of the Planck B-B distribution relates to the flux vector of radiation energy per frequency interval $\Phi(\mathbf{x}, \nu)$ through the equation (83):

$$c\nabla B(\nu, T(\mathbf{x}) = -3\kappa(\mathbf{x}, \nu)\rho(\mathbf{x})\Phi(\mathbf{x}, \nu),$$

b) Show that if we assume radial symmetry equation (83) above becomes equation (86):

$$c\frac{dB(\nu, T(r)}{dr} = -3\kappa(r, \nu)\rho(r)\frac{\mathcal{L}(r, \nu)}{4\pi r^2} \tag{14}$$