Classical Mechanics 2024
Lesson 5: Collisions of particles

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Disintegration of particles

Let's consider a spontaneous disintegration (i.e., atomic fission). We can describe the process in a system where the original particles is at rest.

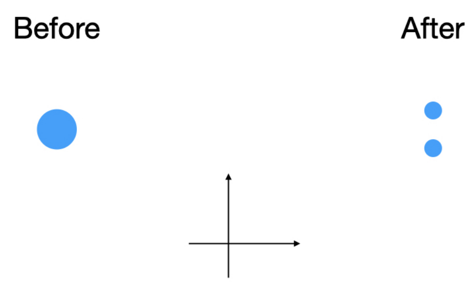

Figure 1: Disintegration of a particle in two.

Momentum and energy are both conserved. At rest the original (one) particle has $\vec{p}_b = 0$. After disintegration the two (let's assume two) resulting particles have $\vec{p}_1$ and $\vec{p}_2$ such $\vec{p}_1 + \vec{p}_2 = 0$.
Then $\vec{p}_1 = -\vec{p}_2$ and $|\vec{p}_1| = |\vec{p}_2| = p_o$. In the system of reference where the system is at rest, the energy is the internal energy E_i before disintegration. Afterwards

$$E_i = E_{1i} + \frac{p_o^2}{2m_1} + E_{2i} + \frac{p_o^2}{2m_2} \tag{1}$$

where m_1 and m_2 are the masses of the resulting particles after disintegration and E_{1i} and E_{2i} their respective internal energies. The disintegration energy is

$$\epsilon = E_i - (E_{1i} + E_{2i}) \tag{2}$$

From (1) and (2) we obtain that

$$\epsilon = \frac{p_o^2}{2m_1} + \frac{p_o^2}{2m_2} = \frac{p_o^2}{2m} \tag{3}$$

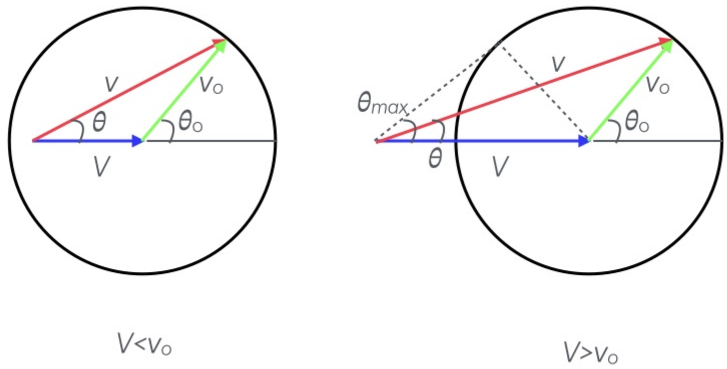

Figure 2: The two possible relations between the angles θ and θ_o for v_o

where $m = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the system.
The velocities of the particles are $v_{10} = \frac{p_o}{m_1}$ y $v_{20} = \frac{p_o}{m_2}$ We can also describe describe the system in a different reference system, one called the Laboratory frame (L), one where the system is not at rest.

Let's label $\vec{V}$ the velocity of the original particle in the L system before the disintegration.
Now let's consider one of the resulting particles: we call $\vec{v}_o$ its velocity as observed from the center of mass system (C), where as observed from the L system it is $\vec{v}$.

The C system has velocity $\vec{V}$ in L. Consequently using Galilean Relativity the relationship between the 2 velocities (the one in the L frame moving at velocity $\vec{V}$ and the other one in the C frame -where the initial particle was at rest- is:

$$\vec{v} - \vec{V} = \vec{v}_o \tag{4}$$

We can square both sides of the equation to obtain the following relation:

$$v^2 + V^2 - 2\ v\ V \cos\theta = {v_o}^2 \tag{5}$$

where θ is the angle between the initial direction of motion of the original particle given by $\vec{V}$ and the direction in which one of the resulting particles moves given by $\vec{v}$. **Notice:** This gives the magnitude of the particle as a function of its direction in the system L.

The figure 2 sketches the three vectors ($\vec{V}$, $\vec{v}$ and $\vec{v}_o$) in two different possible scenarios: one with $V > v_o$, the other one with $V < v_o$. In the first case v_o can have any direction but in the second case the particle can only move forward with a maximum angle θ_{max} given by

$$\sin\theta_{max} = \frac{v_o}{V} \tag{6}$$

The relationship between θ and θ_o is easily made explicit:

$$\tan\theta = \frac{v_o \sin\theta_o}{V + v_o \cos\theta_o} \tag{7}$$

(7) is a quadratic equation for $\cos\theta_o$ (just write the $\sin\theta_o$ as a function of $\cos\theta_o$ and this can be easily seen. The solution is then

$$\cos\theta_o = -\frac{V}{v_o}\sin^2\theta \pm \cos\theta\sqrt{1 - \frac{V^2}{v_o^2}\sin^2\theta} \tag{8}$$

For $v_o > V$ the relation between θ_o and θ is one to one. In this case we take the plus sign in (8) so that $\theta_o = 0$ when $\theta = 0$. But if $v_o < V$ for each value of θ there are two values of θ_o: see figure 2 where the two options are the green vector and another one following the dashed lines from the center of the circle. The values are given by the two signs in equation (8).

Disintegration in many particles

If the disintegration occurs to more than one particle then we need to examine the distribution of all the resulting ones in direction, energy, etc.
We can assume that the primary particles are isotropically (randomly) oriented in space. Using the center of mass reference system C we frame the problem defining stating that the fraction of particles entering an element of solid angle $d\Omega_o$ is precisely proportional to it:

$$d\Omega_o/4\pi \tag{9}$$

Notice that the main parameter describing the spreading of the particles is the angle θ. The distribution of the particles respect to the angle θ_o (remember we are working in the C system!) using that $d\Omega_o = 2\pi\sin\theta_o d\theta_o$ it is from (9)

$$\frac{1}{2}\sin\theta_o d\theta_o \tag{10}$$

If we want to know the corresponding expression in the L system (after all this is the system where observations take place!) we transform appropriately. Let's say we want to know the kinetic energy distribution (in the L system). Solving for v^2 in (5) we get:

$$v^2 = V^2 + {v_o}^2 + 2\ v_o\ V\cos\theta \tag{11}$$

from where we can deduce

$$d\cos\theta = \frac{dv^2}{2\ v_o\ V} \tag{12}$$

The kinetic energy is $T = 1/2\ mv^2$ (the value of m will depend on the particle under consideration). From there we obtain $dT = 1/2\ md(v^2)$. It follow then from (12) that the distribution we want to consider is:

$$d\cos\theta = \frac{dT}{2m\ v_o\ V} \tag{13}$$

The maximum and minimum energy values for the kinetic energy are: $T_{min} = \frac{1}{2}m(v_o - V)^2$ and $T_{max} = \frac{1}{2}m(v_o + V)^2$. When we have several particles resulting from the disintegration of one, momentum and energy are conserved, of course. But there many possible outcomes that would satisfy the conservation laws. Although the energies in particular do not have predetermined values we can calculate an upper limit of them in the C system. Let's concentrate on one of the particles, and we name its mass m_1 and we call the internal energy of the total system of all the other particles but m_1 E_i'. then the kinetic energy of it is

$$T_{10} = p_o^2/2m_1 = \frac{(M - m_1)(E_i - E_{1i} - E_i')}{M} \tag{14}$$

where M is the mass of the original particle before disintegration. We can see that T_{10} can have its largest value when E_i' has the least value. All resulting particles , except for m_1 must be moving with the same velocity. In that case E_i' is just the sum if their internal energies and $E_i - E_{1i} - E_i'$ is the disintegration energy ϵ. Then the maximum kinetic energy that the m_1 particle can have is

$$T_{10,max} = \frac{(M - m_1)\epsilon}{M}. \tag{15}$$

Scattering

If we have an elastic collision between particles, then there is no change in their internal state. These collisions can be characterized by an angle χ which gives the direction between the original momentum of particle 1 and its direction after the collision.
A full calculation of this angle requires solving the equations of motion for the particular nature of the interaction involved.

Let's consider first a similar problem which is the deflection of a single particle of mass m in a field $U(r)$ whose center is at rest and in the centre of mass of the two particles we are considering. Remember, from the chapter on integration of the equations of motion that in central fields the trajectory is symmetric respect to a line from the center of the field to the nearest point in the trajectory (perigee). In the case of the figure 3 it is the line OA. And also let's remember that the trajectories are quadratics involving conic sections. Remembering the symmetry the angle of deflection χ is

$$\chi = |\pi - 2\phi_o| \tag{16}$$

The angle is given by equation (31) from Lesson 3:

$$\phi_o = \int_{r_{min}}^{\infty} \frac{(M/r^2)dr}{\sqrt{2m[E - U(r)] - M^2/r^2}}, \tag{17}$$

where r_{min} is the nearest approach to the center of the field. If we are considering a motion that in practice is infinite it is more convenient to use the energy E and the momentum M as well as the velocity of the particle at infinity v_∞ and the impact parameter ρ.
These are

$$E = \frac{1}{2}mv_\infty^2, \qquad M = m\rho v_\infty \tag{18}$$

With these values (17) becomes

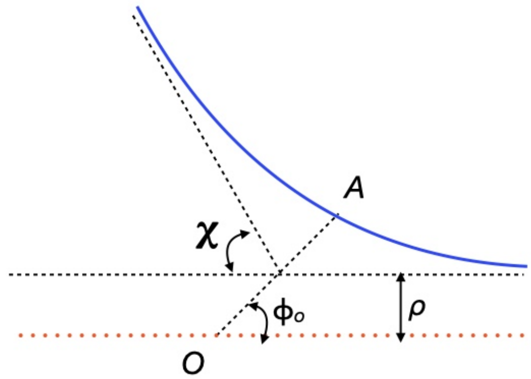

Figure 3: The two relevant parameters in the scattering process

$$\phi_o = \int_{r_{min}}^{\infty} \frac{(\rho/r^2)dr}{\sqrt{1-(\rho^2/r^2)-(2U/mv_\infty^2)}}, \tag{19}$$

(19) together with (16) gives χ as a function of ρ.

In most real life physics applications the goal is not to calculate the deflection angle for a single particle but to measure the scattering of a beam of identical particles launched at uniform velocity v_∞ into a scattering target.

The different particles have different impact parameters and are scattered throughout different angles χ. If dN is the number of particles scattered per unit of time through angles from χ and $\chi + d\chi$ we notice that this number is proportional to the density of the incident beam, ie.

$$n d\sigma = dN \tag{20}$$

where n is the number of particles passing in a unit of time through a differential of area $d\sigma$ of the beam cross-section, where we assume a uniform cross-section. $d\sigma$is called the effective scattering cross-section. It is entirely determined by the nature of the scattering field.

Assuming a one to one relation between χ and ρ (the angle decreases monotonically with the impact parameter). It follows then that $dN = 2\pi\rho d\rho\, n$. The effective cross-section is

$$d\sigma = 2\pi\rho d\rho \tag{21}$$

which can be rewritten in terms of its functionality with χ

$$d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho(\chi)}{d\chi}\right| d\chi \tag{22}$$

where we restrict to the absolute value of the derivative $\left|\frac{d\rho(\chi)}{d\chi}\right|$ because the derivative may be negative. $d\sigma$ is more properly interpreted as an element of solid angle. In this case the solid angle between cones with vertical angles χ and $\chi + d\chi$ is $d\Omega = 2\pi \sin\chi d\chi$ from where we get

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega \tag{23}$$

Rutherford Scattering

The Rutherford scattering experiment revealed that every atom has a nucleus a site or system positively charged concentrating most of its mass. It consisted in measuring how an alpha particle beam was scattered after striking a thin metal foil. The experiments were performed between 1906 and 1913 by Hans Geiger and Ernest Marsden under the direction of Ernest Rutherford at the Physical Laboratories of the University of Manchester.

The physical phenomenon was explained by Ernest Rutherford in a classic 1911 paper that eventually lead to the widespread use of scattering in particle physics to study subatomic matter.Also called Coulomb scattering it is the elastic scattering of charged particles by the Coulomb interaction.

Rutherford scattering is nowadays utilized in the area of materials science as an analytical technique called Rutherford backscattering.

Using $U = \alpha/r$ in equation (19)

$$\phi_o = \int_{r_{min}}^{\infty} \frac{(\rho/r^2)dr}{\sqrt{1-(\rho^2/r^2)-(2\alpha/r\ m\ v_\infty^2)}}, \tag{24}$$

where we can use that

> Remember:
>
> $$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-c}} \operatorname{arcsin} \frac{bx+2c}{x\sqrt{-\Delta}}, \quad X = ax^2+bx+c, \quad \Delta = 4ac-b^2$$

to perform the integration obtaining

$$\phi_o = \sin^{-1} \frac{\alpha/mv_\infty^2\rho}{\sqrt{1+(\alpha/mv_\infty^2\rho)^2}}, \tag{25}$$

from where we obtain $\rho^2 = (\alpha^2/m^2v_\infty^4)\tan^2\phi_o$, and using $\chi = |\pi - 2\phi_o|$

$$\rho^2 = (\alpha^2/m^2v_\infty^4)\cot^2\frac{1}{2}\chi, \tag{26}$$

Deriving respect to χ (which is the relationship we are seeking, $\rho = \rho(\chi)$) and substituting in equation (22) we obtain that the effective cross-section

$$d\sigma = \frac{\pi(\alpha/mv_\infty^2)^2\cos\frac{1}{2}\chi}{\sin^3\frac{1}{2}\chi}d\chi \tag{27}$$

or using (23) in terms of the solid angle we finally obtain the Rutherford formula:

$$d\sigma = \frac{(\alpha/2mv_\infty^2)^2}{\sin^4 \frac{1}{2}\chi} d\Omega \tag{28}$$

Notice that the formula is independent on the sign of α, meaning it is valid for both attractive and repulsive Coulomb fields. The formula above is valid in the C system of the colliding particles.