Classical Mechanics 2024
Lesson 6: Small Oscillations

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Free oscillations in one dimension

Small oscillations of a system about a position of stable equilibrium lead to a modeling of many mechanical systems.

The position of a particle at a minimum of its potential energy corresponds to a situation of stable equilibrium. A slight motion of the particle in any direction immediately generates a restoring force ($-dU/dq$) bringing it back to the equilibrium position. Due to the fact that the potential energy function has its minimum value at its equilibrium position (q_o) we have that $dU/dq(q = q_o) = 0$. So if we make a series expansion of the potential energy function around the equilibrium position we obtain

$$U(q) = U(q_o) + \left.\frac{dU}{dq}\right|_{q=q_o} (q - q_o) + \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_{q=q_o} (q - q_o)^2 + ... \tag{1}$$

where we can keep the first relevant order and write then that the behavior of our potential energy is of the form

$$U(q) - U(q_o) = \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_{q=q_o} (q - q_o)^2 \simeq \frac{1}{2}k(q - q_o)^2 \tag{2}$$

where k is a positive coefficient, the value of the second derivative at $q = q_o$ (why is it positive?). Picking up simpler notation, we can take $U(q_o) = 0$ and use $x = q - q_o$. With these choices we have that

$$U(x) = \frac{1}{2}k\ x^2 \tag{3}$$

The kinetic energy of a system with one degree of freedom is in general of the form $\frac{1}{2}a(q)\dot{q}^2$ In a cartesian coordinate system $a(q)$ can be taken as the mass m and we have then that the Lagrangian for a system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\ x^2 \tag{4}$$

This is a system with just 1 degree of freedom and the Euler Lagrange equation is:

$$m\ddot{x} + kx = 0 \tag{5}$$

which with the following definition

$$\omega = \sqrt{\frac{k}{m}} \tag{6}$$

can be written

$$\ddot{x} + \omega^2 x = 0 \tag{7}$$

which has the general solution

$$x(t) = A\cos\omega t + B\sin\omega t. \tag{8}$$

which can be written as

$$x(t) = \mathcal{A}\cos(\omega t + \alpha) \tag{9}$$

from where we obtain that

$$\mathcal{A} = \sqrt{A^2 + B^2} \tag{10}$$

and

$$\tan\alpha = -\frac{B}{A} \tag{11}$$

Near a position of stable equilibrium a system executes harmonic oscillations. $\mathcal{A}$ is called the amplitude of the oscillations and $\omega t + \alpha$ is called the phase of the oscillations. α is the initial phase, which clearly depends on the choice of initial time. ω is the angular frequency of the oscillations. It is customary to just call it the frequency of the system. Notice that this is a fundamental quantity which does not depend on the initial conditions of motion. Formula (6) indicates that it is related to the fundamental characteristics of the system (i.e. its mass m and physical properties -rigidity or elasticity- k).

Cautionary note: this fact is truth only when the potential energy is a quadratic function of the position of the system. This is not true for **non-small** oscillations.

The energy of a system undergoing small oscillations is:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{12}$$

which using (6) can be written

$$E = \frac{1}{2}m(\dot{x}^2 + \omega x^2) = \frac{1}{2}m\omega^2\mathcal{A}^2 \tag{13}$$

where the last equation derives from formula (10) which defines the amplitude. So the energy is proportional to the square of the amplitude.

The function describing the position of the system as a function of time can also be described in the following manner

$$x(t) = \mathrm{Re}[\mathbb{A}\exp(i\omega t)] \tag{14}$$

where the symbol Re stands for the **real part of**. $\mathbb{A}$ is a complex constant defined as

$$\mathbb{A} = \mathcal{A}\exp(i\alpha)] \tag{15}$$

The complex constant $\mathbb{A}$ is called the complex amplitude. Its magnitude is the regular amplitude, and its argument is the initial phase.

The utilization of the so called Euler's formula takes advantage of the fact that the exponential doesn't change its shape under differentiation.

$$e^{ix} = \cos x + i\sin x \tag{16}$$

In general, assuming that all operations involved are linear we can omit the symbol Re.

Example: the sliding pendulum

See figure 1 for reference. In the following example the system consists of a pendulum of mass m_2 suspended from a mass m_1 which can slide along a rigid horizontal rod. To solve this problem:
1) we will write the Lagrangian for the system (composed of masses m_1 and m_2), then
2) we will integrate the equations of motion for the system, and finally
3) we will calculate the frequency of oscillations of the system.

Solution
1) Of course we assume that this system is subjected to the action of gravity near the Earth's surface (i.e. experiments in a lab). We calculate the Lagrangian of the system as the sum of the Lagrangians for m_1 and m_2 separately. We have that the kinetic energy for mass m_1 is

$$T_1 = \frac{1}{2}m_1\dot{x}^2 \tag{17}$$

The potential energy for m_1 is 0 (why?). Consequently $L_1 = \frac{1}{2}m_1\dot{x}^2$. Regarding mass m_2 its coordinates are:

$$x_2 = x + l\sin\phi, \qquad y_2 = l\cos\phi \tag{18}$$

And then the velocity components are

$$\dot{x}_2 = \dot{x} + l\dot{\phi}\cos\phi, \qquad \dot{y}_2 = -l\dot{\phi}\sin\phi \tag{19}$$

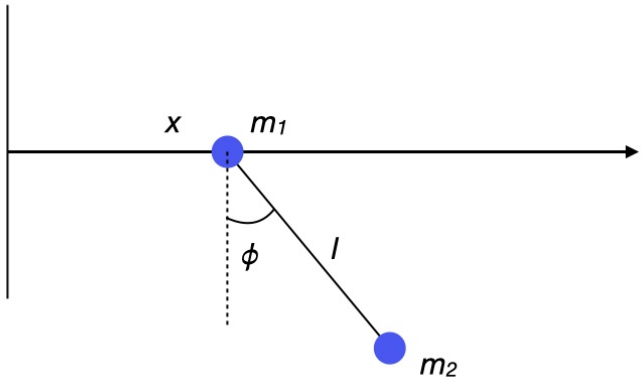

Figure 1: A sliding pendulum.

From where we obtain that

$$T_2 = \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}\dot{\phi}l\cos\phi) \tag{20}$$

and the potential is

$$U_2 = m_2 gl\cos\phi \tag{21}$$

and then the complete Lagrangian is:

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2\dot{x}\dot{\phi}l\cos\phi) + m_2 gl\cos\phi \tag{22}$$

2) In the Lagrangian that we just calculated we can see that the coordinate x does not appear explicitly in the Lagrangian and it is, consequently cyclic. The corresponding E-L equation for x gives:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{23}$$

and then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{24}$$

which is equivalent to

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = C \tag{25}$$

for some constant C where p_x is the momentum of the system in the x direction. This constant is arbitrary and because we can always take the system to be at rest as a whole we can pick the constant to be 0. We then all we need to do is to integrate the equation

$$\frac{dx}{dt} = -\frac{m_2 l\dot{\phi}\cos\phi}{(m_1+m_2)} \tag{26}$$

from where we obtain that

$$(m_1+m_2)x + m_2 l\sin\phi = constant \tag{27}$$

This shows that the horizontal component of the center of mass of the system does not move horizontally.
We will proceed to calculate the energy to use it as the first integral of motion.
We remember that the energy is:

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{28}$$

We then obtain from (22)

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2\dot{\phi}\dot{x}l\cos\phi - m_2 gl\cos\phi \tag{29}$$

Using now (26) to solve to eliminate $\dot{x}$ we obtain

$$E = \frac{1}{2}m_2 l^2\dot{\phi}^2\left(1 - \frac{m_2}{m_1+m_2}\cos^2\phi\right) - m_2 gl\cos\phi \tag{30}$$

And using that E is constant we can integrate to obtain

$$t = l\sqrt{\frac{m_2}{2(m_1+m_2)}}\int\sqrt{\frac{m_1+m_2\sin^2\phi}{E+m_2 gl\cos\phi}}d\phi \tag{31}$$

Notice that equations (18) for x_2 and y_2 coupled with (27) show that the trajectory of mass m_2 is an ellipse with horizontal semi-axis $lm_1/(m_1+m_2)$ and vertical semi-axis l. In equation (30) we can see that for $m_1 \to \infty$ we are left with the traditional pendulum which moves in an arc of circle.

To calculate the frequency of oscillations we assume that $\phi \ll 1$. Just remember that a power series expansion of $\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - ...$ This means that we will keep only $\cos\phi \approx 1 - \frac{\phi^2}{2}$ and neglect higher order terms. Likewise we will take $\cos^2\phi = 1$ for small oscillations. Then the energy E in (30) becomes

$$E = \frac{1}{2}\frac{m_1 m_2 l^2\dot{\phi}^2}{(m_1+m_2)} - m_2 gl + \frac{1}{2}m_2 gl\phi^2 = \frac{1}{2}\frac{m_1 m_2 l^2\dot{\phi}^2}{(m_1+m_2)} + \frac{1}{2}m_2 gl\phi^2 \tag{32}$$

Where $-m_2gl$ is a constant which can be removed by appropriate rescaling of the potential (the potential energy is always defined up to a constant).
Comparing with (12) we identify the effective mass $m = m_1m_2l^2/(m_1 + m_2)$ and the effective constant $k = m_2gl$, rendering

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{(m_1 + m_2)g}{m_1 l}} \tag{33}$$

as the frequency of oscillations for small ϕ.

Forced Oscillations

There are systems where, in addition to a harmonic oscillator potential $U = 1/2kx^2$ (a system undergoing oscillations with only such potential is call a free oscillator), we find also acting upon them a variable external force. The oscillations in such case are called **forced oscillations**. Let's assume that the origin of them is an external field which is weak (we assume still small oscillations). To a first order we can assume that this new field can be written:

$$U_e(x,t) \approx U_e(0,t) + x\frac{\partial U_e}{\partial x}_{|x=0} \tag{34}$$

The first term only depends on time and as such it can be removed from the Lagrangian (it is a total derivative of a function only of time). We can identify then the external force $F_e(t)$ as $F_e(t) = -\frac{\partial U_e}{\partial x}_{|x=0}$ Thus we can write the Lagrangian for such system as

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xF(t) \tag{35}$$

and the associated E-L equation is:

$$m\ddot{x} + kx = F(t) \tag{36}$$

which can also be written

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \tag{37}$$

where ω is the frequency of free oscillations. This is an inhomogeneous differential equation. Its solution is the general solution of the **homogeneous** solution plus a particular solution of the inhomogeneous one. Notice that the homogeneous equation is precisely the one which represents free oscillations. Let's call it x_o and x_1 is a particular one for the inhomogeneous one.

It suffices then to choose one particular solution of the inhomogeneous one. Let's look at a force

$$F(t) = f\cos(\psi t + \beta) \tag{38}$$

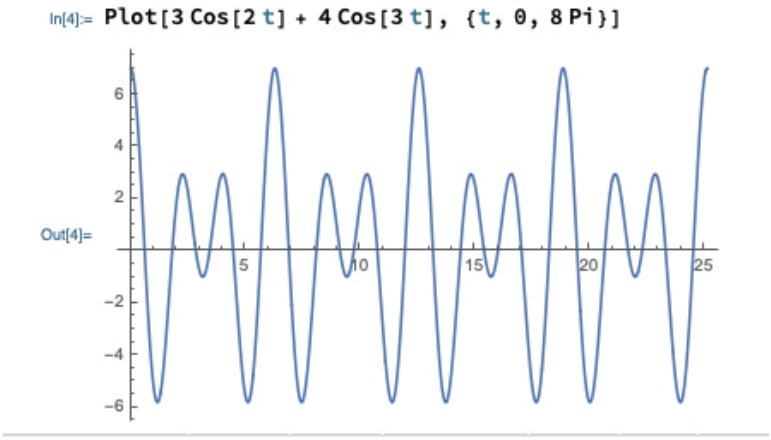

Figure 2: Plot of a system with 2 frequencies of oscillation.

In this case we can look at a solution of the form $x_1(t) = b\cos(\psi t + \beta)$. Notice that the force Plugging it into the differential equation we get that $b = f/m(\psi^2 - \omega^2)$ and then the general solution of the inhomogeneous equation (37) with the force given by (38) is

$$x(t) = A\cos(\omega t + \alpha) + \left[\frac{f}{m(\psi^2 - \omega^2)}\right]\cos(\psi t + \beta) \tag{39}$$

where the value of A and α depend on the initial conditions.

Conclusion the solution consists of a motion with two frequencies ω the intrinsic frequency of the system and another ψ which is the frequency of the external force. In Figure 2 a system with two frequencies is plotted. In this example the initial phases α and β have taken to be 0.

An interesting situation arises if the frequency of the force is the same as the intrinsic frequency of the system. Resonance will occur and the solution (39) is no longer valid. In that case we should rewrite the solution

$$x(t) = A\cos(\omega t + \alpha) + \left[\frac{f}{m(\psi^2 - \omega^2)}\right][\cos(\psi t + \beta) - \cos(\omega t + \beta)] \tag{40}$$

where A is a different amplitude. We can now see what happens if $\psi \to \omega$. We can see that the second term is indeterminate of the form $0/0$. Using L'Hospital rule we obtain

$$x(t) = A\cos(\omega t + \alpha) + \frac{f}{2m\omega}t\sin(\omega t + \beta) \tag{41}$$

which obviously increases in amplitude with time, as can be appreciated in the plot of Figure 3. As

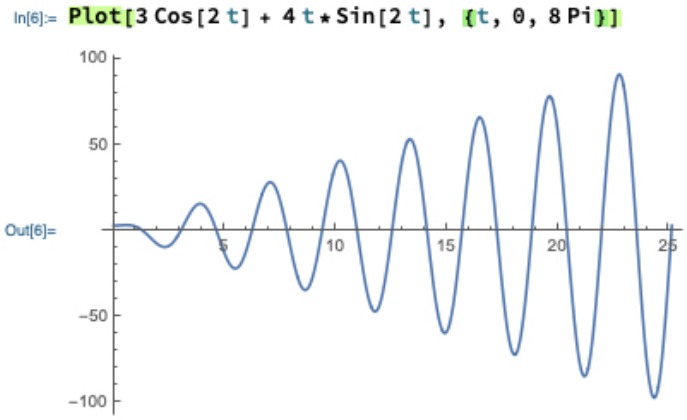

Figure 3: Plot of a system where the forced frequency is the same as the internal one.

the amplitude of the oscillations increase with time we are out of the realm of small oscillations. So the question is: how far can we go assuming small oscillations if we the external force vibrates with a frequency close to the resonance frequency of the system.

To that effect we assume a frequency $\psi = \omega + \epsilon$ where ϵ is small. We express the general solution in complex form

$$x(t) = A\exp(i\omega t) + B\exp[i(\omega+\epsilon)t] = [A + B\exp(i\epsilon t)]\exp(i\omega t) \tag{42}$$

The factor $\exp(i\omega t)$ completes a cycle in a period $2\pi/\omega$. Within it $A + B\exp(i\epsilon t)$ will vary little. Consequently we can consider motion near resonance as small oscillations of variable amplitude (where the amplitude factor is precisely $A + B\exp(i\epsilon t)$. If we call it C then $C = A + B\exp(i\epsilon t)$. We can write $A = a\exp(i\alpha)$ and $B = b\exp(i\beta)$.

Then

$$C = a\exp(i\alpha) + b\exp[i(\beta+\epsilon t)] \tag{43}$$

and calculating the absolute value of C:

$$C^2 = a^2 + b^2 + 2ab\cos(\epsilon t + \beta - \alpha) \tag{44}$$

showing that the amplitude varies with frequency ϵ between the limits

$$|a - b| \leqslant C \leqslant a + b \tag{45}$$

This phenomenon exhibits an interference pattern between two harmonic motions of slightly different frequencies, perceived as a periodic variation in amplitude whose rate -of variation- is the

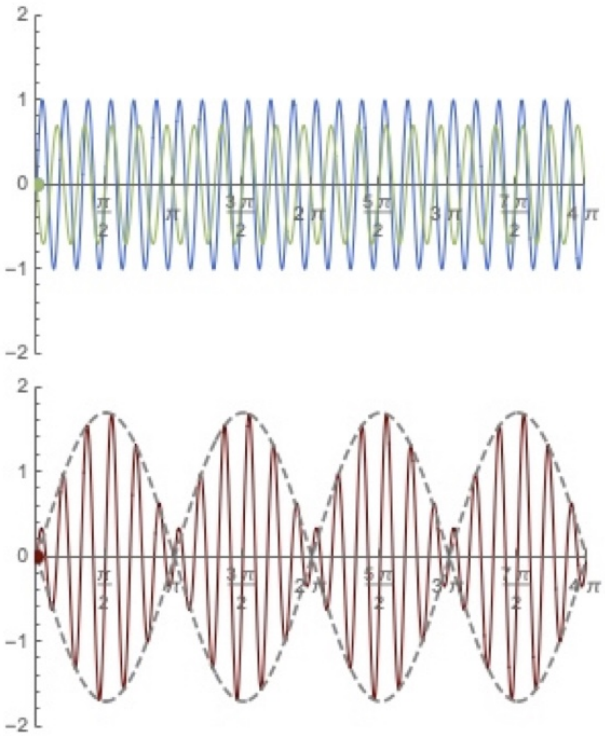

Figure 4: Two harmonic motions. The blue curve has amplitude 1 and frequency 12. The green curve has amplitude 0.7 and frequency 10. In the top graph they are plotted separately. In the bottom their complex sum is plotted.

difference of the two frequencies. In acoustics a beat is precisely the interference pattern between two sounds of slightly different frequencies, perceived as a periodic variation in volume whose rate is the difference of the two frequencies. The figure 4 shows the plot of two harmonic functions of different frequency and amplitude separately and then added. Let's find now a solution of equation (37) where the force is a general arbitrary one $F(t)$. To do this we rewrite it in a complex form as:

$$\frac{d}{dt}(\dot{x} + i\omega x) - i\omega(\dot{x} + i\omega x) = \frac{1}{m}F(t) \tag{46}$$

We can define then $\xi = \dot{x} + i\omega x$ converting the above equation in

$$\frac{d\xi}{dt} - i\omega\xi = \frac{F(t)}{m} \tag{47}$$

This is a first order equation which has for the homogeneous form $\xi = A\exp(i\omega t)$ with A constant. To find just one solution of the inhomogeneous equation we choose $\xi = A(t)\exp(i\omega t)$. When plug

into (47) converting it into an equation for $A(t)$:

$$\dot{A} = \frac{F(t)}{m} \exp(-i\omega t) \tag{48}$$

From this

$$A = \int_0^t \frac{F(t)}{m} \exp(-i\omega t) dt \tag{49}$$

And then

$$\xi = \exp(i\omega t) \left\{ \int_0^t \frac{F(t)}{m} \exp(-i\omega t) dt + \xi_0 \right\} \tag{50}$$

ξ_0 is the value of ξ at $t = 0$. (50) is the general solution of (47). From there we can obtain $x(t)$ just using the definition of ξ, namely $\xi = \dot{x} + i\omega x$.

On a different note we can calculate the energy transmitted to the system by the external force during all time. Using (50) where we take $t_0 = -\infty$ with $\xi(-\infty) = 0$ wen $t \to \infty$

$$|\xi(\infty)|^2 = \frac{1}{m^2} \left| \int_{-\infty}^{\infty} F(t) \exp(-i\omega t) dt \right|^2 \tag{51}$$

The energy of the system is

$$E = \frac{1}{2}(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2} m |\xi|^2. \tag{52}$$

from where using (51) we obtain the energy transferred during all the time that the action of the external field lasted.

$$E = \frac{1}{2m} \left| \int_{-\infty}^{\infty} F(t) \exp(-i\omega t) dt \right|^2 \tag{53}$$

If the time is short compared with $1/\omega$, $\exp(-i\omega t) \cong 1$ and then

$$E = \frac{1}{2m} \left(\int_{-\infty}^{\infty} F(t) dt \right)^2 \tag{54}$$

Modeling of multidimensional physical systems

The theory developed so far extended to systems with several degrees of freedom is at the foundation of the treatment of many real physical systems in acoustics, molecular spectrography, vibrations in general and coupled electrical systems.

Let's extend the theory we just studied to the case of multiple degrees of freedom. We assume that the system, which has n degrees of freedom and is described by n generalized coordinates q_i is such that its potential energy has a minimum at $q_i = q_{i0}$.

As we did before we expand the potential up to terms quadratic in $x_i = q_i - q_{i0}$. The result will be a potential energy

$$U(x_i) = \frac{1}{2}\sum_{i,l} k_{il} x_i x_l \tag{55}$$

where the sum is taken over the two indices. Due to the fact that the potential is symmetric around its minimum for displacements from it small enough, its fair to assume $k_{il} = k_{li}$.

In Lesson 1, formula (57) we showed how using generalized coordinates, which are in general different from cartesian ones, the general formula for the kinetic energy contains a coefficient multiplying the generalized velocities of the form $a_{i,l}(q)$ as in

$$\frac{1}{2}\sum_{i,l} a_{i,l}(q)\dot{q}_i\dot{q}_l \tag{56}$$

As we are considering only terms quadratic in q we can take $a_{il}(q_i) = a_{il}(q_0) = m_{il}$ and obtain for the kinetic energy

$$\frac{1}{2}\sum_{i,l} m_{il}\dot{x}_i\dot{x}_l \tag{57}$$

where we consider $m_{il} = m_{li}$. The Lagrangian then for a system undergoing free oscillations is:

$$L = \frac{1}{2}\sum_{i,l}(m_{il}\dot{x}_i\dot{x}_l - k_{il}x_i x_l) \tag{58}$$

We cam simplify the calculation of the E-L equations just calculating the differential of L.

$$dL = \frac{1}{2}\sum_{i,l}(m_{il}\dot{x}_i d\dot{x}_l + m_{il}\dot{x}_l d\dot{x}_i - k_{il}x_i dx_l - k_{il}x_l dx_i) \tag{59}$$

The suffices are dummy indices. We can exchange them and use the symmetry of m_{il} and k_{il} getting

$$dL = \sum(m_{il}\dot{x}_i d\dot{x}_l - k_{il}x_l dx_i) = \sum\left(\frac{\partial L}{\partial \dot{x}_i}d\dot{x}_i - \frac{\partial L}{\partial x_i}dx_i\right) \tag{60}$$

and then

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_l m_{il}\dot{x}_l \qquad \frac{\partial L}{\partial x_i} = -\sum_l k_{il}x_l \tag{61}$$

All we are left to do now is to calculate the total time derivative of the first term and the the E-L equations are

$$\sum_l m_{il}\ddot{x}_l - \sum_l k_{il}x_l = 0 \qquad l = 1,\ 2, ...n; \tag{62}$$

These are n linear second order homogeneous differential equations where the coefficients are constant. The solution can be sought as before

$$x_l = A_l \exp(i\omega t) \tag{63}$$

We plug in (63) in (62) and get

$$\sum_l \left(-\omega^2 m_{il} + k_{il}\right) A_l = 0 \tag{64}$$

Notice that A_l is a vector and $k_{il} - \omega^2 m_{il}$ a matrix. (64) is a system of n algebraic equations. If it has non trivial solutions (i.e. non zero) the necessary and sufficient condition is that the determinant of $k_{il} - \omega^2 m_{il}$ satisfies

$$\sum_l |k_{il} - \omega^2 m_{il}| = 0 \tag{65}$$

In linear algebra, eigenvalues and eigenvectors play a fundamental role, since, given a linear transformation, an eigenvector is a vector whose direction is not changed by the transformation, and the corresponding eigenvalue is the measure of the resulting change of magnitude of the vector.
More precisely, if the transformation is represented by a square matrix A an eigenvector $\vec{V}$ and the corresponding eigenvalue is λ it must satisfy the equation $A\vec{V} = \lambda\vec{V}$ equivalent to $(\lambda I - A)\vec{V} = 0$ Where I is the identity matrix and we require $\vec{V} \neq 0$ From where it follows that the matrix $\lambda I - A$ must be singular (or degenerate or non-invertible). For this the necessary and sufficient condition is that

$$det\,(\lambda I - A) = 0$$

λ_i are the roots of the called characteristic polynomial $det\,(xI - A)$ which is of degree n is that is the order of the matrix. These are called the eigenvalues of the eigenvector $\vec{V}$.

(65) then is the characteristic equation, which is of degree n in ω^2. In general it has n different real positive roots ${\omega_i}^2$ where $i = 1,\ 2, ...n$. It may be the case that some are the same. The quantities determined by this equation ω_i are the characteristic frequencies of the system (also called *eigenfrequencies*).

The roots of (65) are positive for both physical and mathematical considerations. Once we find the frequencies solving (65) we substitute them in (64) to find A_l. If all the roots ω_α are different (no repeated eigenvalues), the coefficients A_l are proportional to the minors of the determinant (65) with $\omega = \omega_\alpha$.

> The minor of a given element of a matrix (for a square one of course) is the part of the matrix remaining after excluding the row and the column containing the element.

If we call these minors Δ_{kl} then a particular solution of (62) is

$$x_k(t) = \Delta_{k\alpha} C_\alpha \exp(i\omega_\alpha t) \tag{66}$$

where the coefficient C_α is an arbitrary complex constant. The general solution is just the sum of all the n particular solutions. Of course we will take the real part of it.

$$x_k(t) = \mathrm{Re} \sum_{\alpha=1}^{n} \Delta_{k\alpha} C_\alpha \exp(i\omega_\alpha t) = \sum_{\alpha} \Delta_{k\alpha} \Psi_\alpha \tag{67}$$

where

$$\Psi_\alpha = \mathrm{Re}\left[C_\alpha \exp(i\omega_\alpha t)\right] \tag{68}$$

This means that the time variation of each coordinate of the system (near equilibrium) can be represented as a superposition of n periodic oscillations $\Psi_1, \Psi_2, ...\Psi_n$ with frequencies given by $\omega_1, \omega_2, ...\omega_n$ and arbitrary amplitudes and phases (which depend on the initial conditions).

Normal Coordinates

Can the generalized coordinates be chosen so that each of them undergoes one simple oscillation? The answer resides precisely in (67). (67) shows that given a set or arbitrary coordinates we need to find another set of Ψ_i which can be expressed as functions of the original generalized coordinates such that the solution is precisely a linear combination of them (the eigenvectors). These new coordinates also called the *normal* coordinates express the motion of the system as a set of simple independent oscillations [1].

These normal coordinates satisfy by construction

[1] The topic of diagonalization of matrices and the eigenvalues and eigenvectors problems in the context of ordinary differential equations can be reviewed quickly in for example https://math.mit.edu/ jorloff/suppnotes/suppnotes03/la5.pdf or https://en.wikipedia.org/wiki/Principal_axis_theorem

$$\ddot{\Psi}_\alpha + {\omega_\alpha}^2 \Psi_\alpha = 0 \tag{69}$$

The normal oscillations are thus independent. The transformation (67) diagonalizes both the kinetic energy and the potential energy. Therefore the Lagrangian is

$$L = \sum \alpha \frac{1}{2} m_\alpha \left(\dot{\Psi}_\alpha^2 - {\omega_\alpha}^2 {\Psi_\alpha}^2 \right), \tag{70}$$

where m_α are positive constants. We have diagonalized both the kinetic and potential energy. We can define a new set of coordinates where we can eliminate the constants m_α through an appropriate definition.

$$Q_\alpha = \sqrt{m} \Psi_\alpha \tag{71}$$

making the Lagrangian then

$$L = \frac{1}{2} \sum_\alpha (\dot{Q}_\alpha^2 - \omega^2 Q_\alpha^2), \tag{72}$$

The treatment of the situation is similar in the case of repeated frequencies (so called degenerate frequencies). In that case the solution will be a linear combination of normalized coordinates which is not unique (there are infinitely many possible combinations as a result of the degeneracy).

Three dimensional Oscillator in a constant external field

We choose the origin of Cartesian coordinates at the minimum value of the potential $U(x, y, z)$. The Lagrangian will be

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}(k_1 x^2 + k_2 y^2 + k_3 z^2) \tag{73}$$

and the normal oscillations take place in the x, y, z directions with frequencies $\omega_1 = \sqrt{k_1/m},\ \omega_2 = \sqrt{k_2/m},\ \omega_3 = \sqrt{k_3/m}$. In the particular case of a central field we have that the three frequencies are the same and $U = (1/2)kr^2$.

Forced Oscillations with more than one degree of freedom

The formalism just used above makes it simple to generalize the analysis to the case of forced oscillations in several dimensions.
The Lagrangian of the system can be described as

$$L = L_0 + \sum_k F_k(t) x_k \tag{74}$$

where L_0 is the Lagrangian corresponding to free oscillations. Changing to a system of normal coordinates we get

$$L = \frac{1}{2}\sum_{\alpha}(\dot{Q}_{\alpha}^2 - \omega^2 Q_{\alpha}^2) + \sum_{\alpha} f_{\alpha}(t)Q_{\alpha}, \tag{75}$$

where

$$f_{\alpha}(t) = \sum_{k}(t)\frac{\Delta_{k\alpha}}{\sqrt{m}}, \tag{76}$$

and the corresponding equations of motion are

$$\ddot{Q}_{\alpha} + \omega^2 Q_{\alpha} = f_{\alpha}(t), \tag{77}$$

where each equation involves only one unknown $Q_{\alpha}(t)$.

Free vibrations of a linear triatomic molecule

Let's look at a linear triatomic molecule (see Figure 5). Two masses m are located at each side of another mass M at an equilibrium distance b from it. Let's consider vibrations in the same direction. We will approximate the molecular forces as springs of constant k. The equilibrium positions for each i particle are x_{0i}

$$V = \frac{k}{2}(x_2 - x_1 - b)^2 + \frac{k}{2}(x_3 - x_2 - b)^2 \tag{78}$$

To simplify notation we can define

$$\eta_i = x_i - x_{0i} \tag{79}$$

Notice that

$$x_{02} - x_{01} = b = x_{03} - x_{02} \tag{80}$$

Then the potential can be written

$$V = \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2 \tag{81}$$

which can be written

$$V = \frac{k}{2}(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3) \tag{82}$$

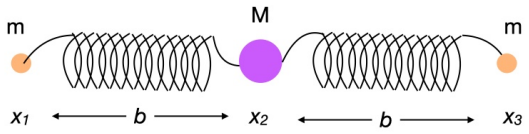

Figure 5: A simple triatomic molecule

The three independent generalized coordinates η_1, η_2, η_3 can be represented as a vector and the elastic constants as a tensor in the following manner

$$V = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \end{pmatrix} \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \tag{83}$$

The kinetic energy is simply

$$T = \frac{m}{2}(\dot{\eta_1}^2 + \dot{\eta_3}^2) + \frac{M}{2}\dot{\eta_2}^2 \tag{84}$$

Where the associated mass tensor is:

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \tag{85}$$

Notice that the problem can be posed as the diagonalization of the matrix V in the eigenvalue equation

$$V - \omega^2 T \tag{86}$$

Its determinant being made identically 0 gives us the secular equation:

$$det(V - \omega^2 T) = |V - \omega^2 T| = \begin{vmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{vmatrix} = 0 \tag{87}$$

The determinant being equal to 0 gives a cubic equation in (ω^2)

$$\omega^2(k - \omega^2 m)\left\{k(M + 2m) - \omega^2 M m\right\} = 0 \tag{88}$$

It is easy to identify the roots as $\omega_1 = 0$, $\omega_2 = \sqrt{k/m}$ and $\omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$.
Obviously $\omega_1 = 0$ corresponds to $\ddot{\eta_1} = 0$ which corresponds to a uniform translational motion (no

vibration). There are not "three" degrees of freedom for vibrational motion: one of them is a rigid body degree of freedom.

Damped oscillations

To a great extent we have not considered frictional forces in this course. In real life, frictional forces are always present. Motion does not take place in a vacuum.

If the frequencies of oscillation are small compared to the ones involving dissipative forces we can consider that the net effect is the existence of a frictional force that depends on the medium (which we consider homogeneous) and of the velocity of the oscillating body. If the velocity is small the frictional force can be expanded in powers of the velocity. The zero order of course corresponds to the body at rest, and then the first relevant order is linear in the velocity. We can then write the frictional force as $f_f = \alpha \dot{x}$.

The equation of motion will be then:

$$m\ddot{x} = -kx - \alpha\dot{x} \tag{89}$$

which defining $\omega_0^2 = k/m$, the frequency of free oscillations and $\alpha/m = 2c$ where c is called the damping coefficient, becomes

$$\ddot{x} + 2c\dot{x} + \omega_0^2 x = 0 \tag{90}$$

We want to seek a solution of the form $x = \exp(\lambda t)$. Notice that (90) becomes then,

$$\lambda^2 x + 2c\lambda x + \omega_0^2 x = 0$$

which of course yields a quadratic equation in λ

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0 \tag{91}$$

The solution is $\lambda_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$ which gives us the general solution of (90) as:

$$x(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t) \tag{92}$$

There are two possible cases:

1) $c < \omega_0$

In this case λ is complex and there two complex conjugates values for it. The soultion can be written as:

$$x(t) = \Re e\left[A\exp\left(-ct + \imath\sqrt{(\omega_0^2 - c^2)}t\right)\right] \tag{93}$$

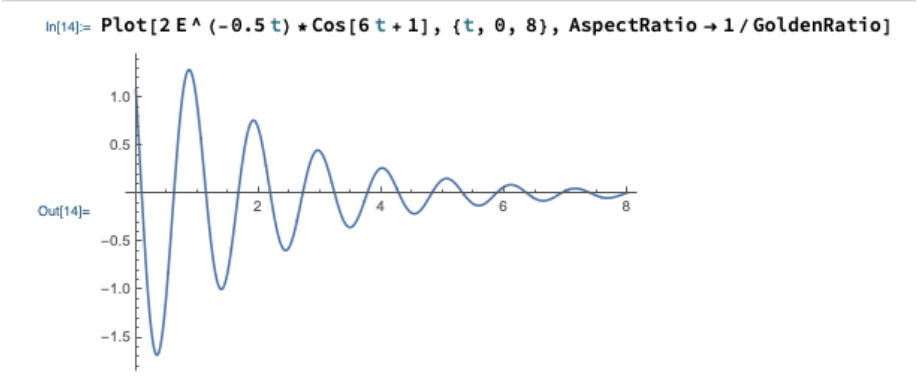

Figure 6: A damped oscillator, plotted using Mathematica function Plot.

where A is an arbitrary complex constant, or as

$$x(t) = a\exp(-ct)\cos\left(\left(\sqrt{\omega_0^2 - c^2}\right)t + \alpha\right) \tag{94}$$

where a and α are real constants. The motion described by (94) is called a damped oscillation. A typical plot can be seen in Figure 6.

As it can be seen for the particular example of the plot (where $a = 2$, $c = 0.5$, $\sqrt{\omega_0^2 - c^2} = 6$, $\alpha = 1$ -how much is c?-) these are harmonic oscillations of exponentially decreasing amplitude. Notice that the frequency of oscillations $\sqrt{\omega_0^2 - c^2}$ is less than the free oscillations ω_0 which would correspond to $c = 0$.
If $c \ll \omega_0$ the difference between $\sqrt{\omega_0^2 - c^2}$ and ω_0 is of second order (see Figure 7 and compare with Figure 6).

As it can be seen in the plot of Figure 7 the damped oscillation is almost unchanged over the period $2\pi/\sqrt{\omega_0^2 - c^2}$. When calculating the average energy, which is of the form $(1/2)m\dot{x}^2 + (1/2)kx^2$, over a period of oscillation, they will be proportional to $\exp(-ct)$. This implies that the mean energy of the system is decreasing, due to friction, as

$$E = E_0\exp(-2ct) \tag{95}$$

where E_0 is the initial value of the energy.

2) $c > \omega_0$

The values of λ are both real and negative.

$$x(t) = A\exp\left\{\left[-c - \left(\sqrt{c^2 - \omega_0^2}\right)\right]t\right\} + B\exp\left\{\left[-c + \left(\sqrt{c^2 - \omega_0^2}\right)\right]t\right\} \tag{96}$$

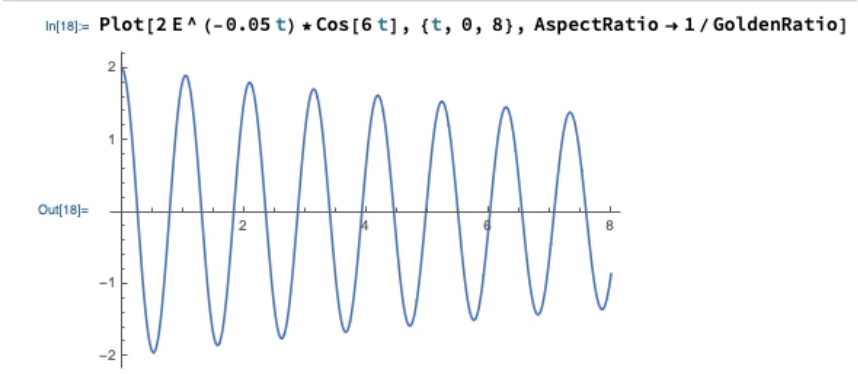

Figure 7: A damped oscillator, with c lower than in Figure 6.

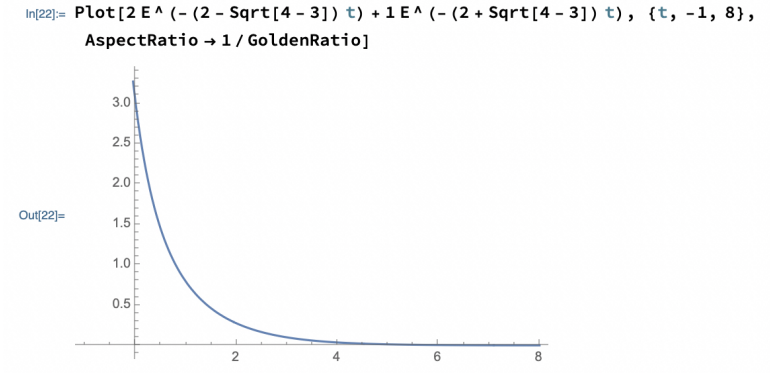

Figure 8: Aperiodic damping.

An example of this type of motion, called aperiodic damping is plotted in Figure 8 where $A = 2$, $B = 1$, $c = 2$, and $\omega_0^2 = 3$. And finally, the last case,

3) $c = \omega_0$

Then $\lambda = -c$ and the general solution is

$$x = (A + Bt)\exp(-ct) \tag{97}$$

This is a particular case of aperiodic damping (see Figure 9).

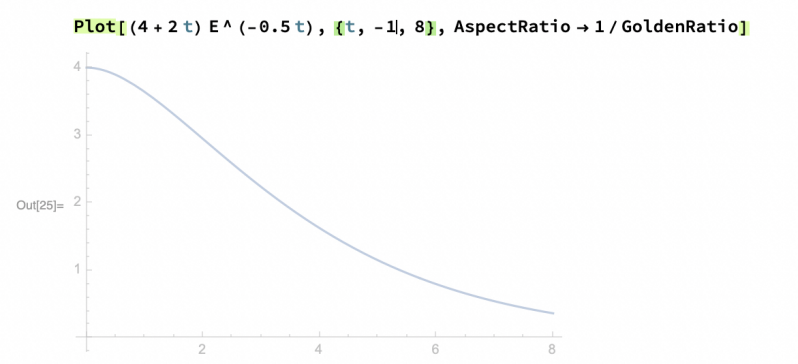

Figure 9: A particular case of aperiodic damping.

Parametric resonance

Parametric resonance is the resonance induced in a non-closed system when one of the system parameters is a function of time and its external variation allows for the flow of energy into the system. A good example is an induced change in the length of a suspended pendulum. We can assume as in the rest of this lesson a mass m and a "force" constant k, where for example $m = m(t)$

$$\frac{d}{dt}(m\dot{x}) + kx = 0 \tag{98}$$

With a change of variables $d\tau/dt = 1/m(t)$

$$\frac{1}{m}\frac{d}{d\tau}(m\frac{d}{md\tau}x) + kx = 0$$

which it is obviously

$$\frac{d^2x}{d\tau^2} + \omega(\tau)x = 0 = \frac{d^2x}{dt^2} + \omega(t)x \tag{99}$$

and where $\omega(t) = m(t)k$. This function characterizes the problem. We can assume that it is a function with period T and frequency $\gamma = 2\pi/T$. This implies that $\omega(t+T) = \omega(t)$, so that equation (99) is invariant under the transformation $t \to t+T$. If $x(t)$ is a solution so it is $x(t+T)$. Let's assume that $x_1(t)$ and $x_2(t)$ are two independent solutions of this second order equation. Then the most general solution of (99) is a sum of both equations with constant coefficients which satisfy the initial conditions. Notice that this means that if we replace t by $t+T$ It is possible to

choose a combination of x_1 and x_2 equivalent to a diagonalization of the system. In particular we can choose them $x_1(t+T) = \mu_1 x_1(t)$ and $x_2(t+T) = \mu_2 x_2(t)$. The most general functions having such properties are

$$x_1(t) = \mu_1^{t/T} H_1(t), \qquad x_2(t) = \mu_2^{t/T} H_2(t), \tag{100}$$

where $H_1(t)$ and $H_2(t)$ are periodic functions of t of period T.
How are μ_1 and μ_2 related? We can multiply $\ddot{x}_1 - \omega^2(t)x_1 = 0$ by x_2 and $\ddot{x}_2 - \omega^2(t)x_2 = 0$ by x_1. We then obtain that

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = constant \tag{101}$$

If in the expression above t is replaced by $t+T$ we will get an overall factor $\mu_1\mu_2$ that needs to be 1 for it to remain true.

$$\mu_1\mu_2 = 1 \tag{102}$$

If we assume that the coefficient in (99) is real we can obtain extra information for $\mu_1\mu_2$. If $x(t)$ is an integral of (99) then $x^*(t)$, its complex conjugate must also be an integral. This implies that either $\mu_i = \mu_i^*$ $i = 1, 2$ in the posible combinations, or they are both real. In all case it's clear that $|\mu_1|^2 = |\mu_2|^2 = 1$. If they are both real and the two independent integrals of (99) are

$$x_1(t) = \mu^{t/T} H_1(t), \qquad x_2(t) = \mu^{-t/T} H_2(t), \tag{103}$$

with a positive or negative real value of μ ($|\mu| \neq 1$) One of either function depending on $|\mu| > 1$ or $|\mu| < 1$ increases exponentially with time. The system resting position ($x = 0$) is unstable. Any minimal displacement will increase exponentially with time: this is what is called **parametric resonance**.
one of the differences with ordinary resonance is that if $x(t=0) = 0$ and $\dot{x}(t=0) = 0$ the system remains at rest.

Parametric resonance when the physical parameter departures slightly from a constant value

Let's assume that

$$\omega^2(t) = \omega_0^2(1 + h\cos\gamma t) \tag{104}$$

where ω_0 is constant and $h \ll 1$, For reason that will become clear later let's pick $\gamma = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$ The equation of motion is now:

$$\frac{d^2}{dt^2}x + \omega_0^2[1 + h\cos(2\omega_0 + \epsilon)]tx = 0 \tag{105}$$

This is a second order differential equation. This differential equation is called a Mathieu's equation. It is a linear differential equation with variable coefficients (which vary with a given period). It commonly occurs in nonlinear oscillation problems. (For an interesting discussion and examples see [2]).

A solution could be found of this form:

$$x(t) = a(t)\cos(\omega_0 + \frac{1}{2}\epsilon)t + b(t)\sin(\omega_0 + \frac{1}{2}\epsilon)t, \tag{106}$$

where $a(t)$ and $b(t)$ vary slower than $\cos$ and $\sin$. As we plug it in we could see that this is not an exact solution. But we want to keep only terms of first order in h. For example terms that are of higher frequencies like $2(\omega_0 + 1/2\epsilon)$ may be neglected. We substitute (106) in (105) and keep only terms up to first order in ϵ and expect that the involved time derivatives behave like $\dot{a} \sim \epsilon a$ and $\dot{b} \sim \epsilon b$.

Where we have products of the form $\cos(\omega_0 + \frac{1}{2}\epsilon)t \cdot \cos(2\omega_0 + \epsilon)t$ can be replaced by the sum:

$$\cos(\omega_0 + \frac{1}{2}\epsilon)t \cdot \cos(2\omega_0 + \epsilon)t = \frac{1}{2}\cos 3(\omega_0 + \frac{1}{2}\epsilon)t + \frac{1}{2}\cos(\omega_0 + \frac{1}{2}\epsilon)t \tag{107}$$

and similarly, omitting terms which are $3(\omega_0 + \frac{1}{2}\epsilon)$ and higher. The result is:

$$-(2\dot{a} + b\epsilon + \frac{1}{2}h\omega_0 b)\omega_0 \sin(\omega_0 + \frac{1}{2}\epsilon)t + (2\dot{b} - a\epsilon + \frac{1}{2}h\omega_0 a)\omega_0 \cos(\omega_0 + \frac{1}{2}\epsilon)t = 0 \tag{108}$$

But this is a linear combination of $\sin$ and $\cos$ equal to 0. It could only be possible if the coefficients are also 0.
this then gives us two linear differential equation of first order for $a(t)$ and $b(t)$:

$$2\dot{a} + (\epsilon + \frac{1}{2}h\omega_0)b = 0 \tag{109}$$

$$2\dot{b} - (\epsilon - \frac{1}{2}h\omega_0)a = 0 \tag{110}$$

We will seek solutions of the form $a(t) \propto e^{st}$. This leads to:

$$sa + \frac{1}{2}(\epsilon + \frac{1}{2}h\omega_0)b = 0 \tag{111}$$

$$sb - \frac{1}{2}(\epsilon - \frac{1}{2}h\omega_0)a = 0 \tag{112}$$

posing immediately the following compatibility equation:

$$s^2 = \frac{1}{4}\Big[(\frac{1}{2}h\omega_0)^2 - \epsilon^2\Big] \tag{113}$$

[2] http://audiophile.tam.cornell.edu/randpdf/rand_mathieu_CISM.pdf

Note that s is related to μ in (103) by $\mu = -e^{s\pi/\omega_0}$. We expect then s to be real (i.e. $s^2 > 0$), so

$$\frac{1}{2}h\omega_0 > \epsilon > -\frac{1}{2}h\omega_0. \tag{114}$$

This is a range for parametric resonance on either side ot the frequency $2\omega_0$. As it can be seen the range of resonance can be quickly obtained by making $s = 0$ and then $\epsilon = \pm\frac{1}{2}h\omega_0$. The width of the range is proportional to h. Parametric resonance also occurs when the frequency γ with which the parameter varies is close to any value $2\omega_0/n$ with n integer. The width of the range of resonance will decrease rapidly with increasing n. And the amplification coefficient for the oscillation will also decrease. In the case of the existence of frictional forces parametric resonance will still exist but the region of instability will be narrowed down in relation to the damping forces. Damping goes as $e^{-\lambda t}$. Parametric resonance will be damped as $e^{(s-\lambda)t}$ where s is the solution corresponding to the frictionless case. The limit of the region of instability os determined by $s - \lambda = 0$. With s given by (113) the range of resonance with friction will be given by

$$-\sqrt{(\frac{1}{2}h\omega_0)^2 - 4\lambda^2} < \epsilon < \sqrt{(\frac{1}{2}h\omega_0)^2 - 4\lambda^2}. \tag{115}$$

instead of (114). Resonance is now only possible when h exceeds a threshold value $h_k = 4\lambda/\omega_0$.

Motion in a rapidly oscillating field

We will consider now the motion of a particle in the field of a time independent field (like we have been doing so far) with the addition of a force

$$f = f_1 \cos\omega t + f_2 \sin\omega t. \tag{116}$$

As it is easily noticed this is a periodic time varying force. We will assume that ω, the frequency, is high (so a rapidly oscillating field). How high? a good measure is the frequency of motion of the particle in the time independent field alone. Assuming that the period of that motion without the force (116) is T then we assume that $\omega \gg 1/T$. At the same time we will assume that the amplitude ψ of the oscillations experienced in the motion of our particle due to (116) is small, although the magnitude of it is not necessarily considered small compared to the force resultant of the action of U. We will simplify our model considering 1-dimensional motion dependent of the cartesian coordinate x. The equation of motion in this case then is

$$m\ddot{x} = -\frac{dU}{dx} + f \tag{117}$$

We can safely assume due to the linearity of the equation that the motion will combine a smooth trajectory due to U and an oscillatory motion due to f. So we assume $x(t)$ is of the form

$$x(t) = X(t) + \psi(t), \tag{118}$$

Let's remember our claim that $\omega \gg 1/T$ where T is the period of the unperturbed motion. The mean value of $\psi(t)$ over $2\pi/\omega$ is of course zero. And in that time $X(t)$ will change little. Then the average motion of the particle is $\overline{x} = X(t)$ where the overall average is precisely the motion the particle would have taken if unperturbed by f. Substituting (118) in (117) and expanding in powers of ψ up to first order

$$m\ddot{x} + m\ddot{\psi} = -\frac{dU}{dx} - \psi\frac{d^2U}{dx^2} + f(X,t) + \psi\frac{\partial f}{\partial X} \tag{119}$$

Notice again, that due to our assumption of a linearly superimposed effect the equation involve the oscillatory and the undisturbed terms separately. It is natural to to have them separately equal then.

$$m\ddot{\psi} = f(X,t); \tag{120}$$

The other terms will include the oscillatory displacements ψ and are smaller in higher orders. Although we need to consider that $\ddot{\psi} \propto \omega^2$ which due our original assumption is not small. Integrating (120) with f given by (116) we get

$$\psi = -\frac{f}{m\omega^2}. \tag{121}$$

Next we calculate the time average of (119) respecting the considerations we made before. The implications are that the mean values of the first powers of f and ψ are zero, so we get

$$m\ddot{X} = -\frac{dU}{dX} + \overline{\psi\frac{\partial f}{\partial X}} = -\frac{dU}{dx} - \frac{1}{m\omega^2}\overline{f\frac{\partial f}{\partial X}}, \tag{122}$$

a result which involves only $X(t)$. The "effective" equation is

$$m\ddot{X} = -\frac{dU_{eff}}{dX}, \tag{123}$$

where the "effective" potential energy is

$$U_{eff} = U + \frac{\overline{f^2}}{2m\omega^2} = U + \frac{f_1^2 + f_2^2}{4m\omega^2}. \tag{124}$$

It is appropriate to note that when calculating the average of the time derivative of ψ from (121) we obtain

$$\overline{\dot{\psi}} \propto \frac{\overline{f}}{m\omega}. \tag{125}$$

thus the final result express the fact that the added term in (124) is the mean kinetic energy of the oscillatory motion.

$$U_{eff} = U + \frac{1}{2}m\overline{\dot{\psi}^2}. \tag{126}$$

The result we obtained here is that the motion of the particle averaged over the oscillations is the same as if the potential U was augmented by a quantity proportional to the squared of the amplitude of the oscillatory field. This result can be generalized for an arbitrary number of degrees of freedom using generalized coordinates. Of course the coefficients of the average amplitude of the oscillatory field are not necessarily the mass, although there are the coefficients of the square of the generalized velocities in the kinetic energy.