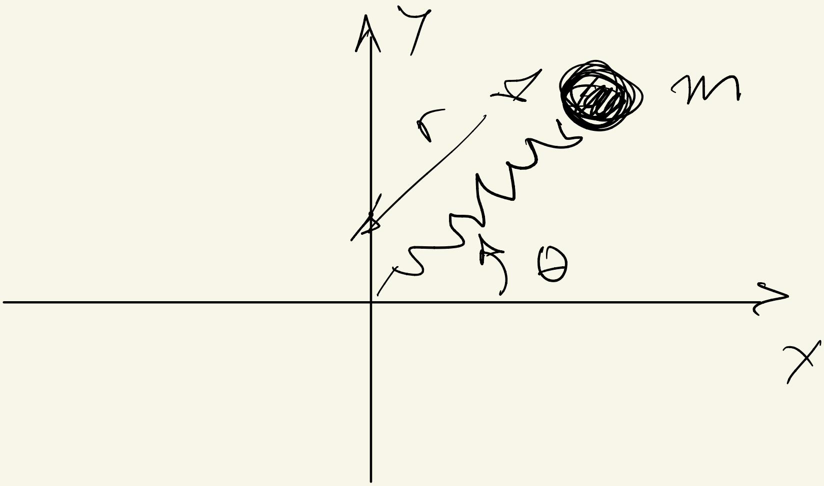
Lesson 9



$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \frac{1}{2} k r^2$$

It is easy to see that the system is time invariant and also it does not depend on θ.

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} k r^2$$

θ being cyclic

$$M = m r^2 \dot{\theta}$$

momentum is conserved

To calculate H we switch to
to r, p_r, θ, p_θ

$p_\theta = mr^2\dot{\theta} = M$ and $p_r = m\dot{r}$

$$\rightarrow \quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2 = E$$

Some <u>definitions</u>:

<u>Involution</u>

two functions F_1 and F_2 are in <u>involution</u> if their Poisson bracket is 0.

$$[F_1, F_2] = 0$$

<u>Integrable systems</u>

If a system with N degrees of freedom has N conserved quantities that are in involution then the system is integrable i.e.

If the conserved quantities of the system are denoted by

$$C_i = C_i(q, p)$$

$i = 1, \dots, N$, each being a function of all the $2N$ phase-space coordinates

Then the system is integrable if

$$[C_i, C_j] = 0 \quad \forall\, i, j$$

Let's apply it to our example

We can think that the C_is are H and p_θ for example.

these are 2 quantities that are conserved.

We calculate the Poisson brackets

$$[H, p_\theta] = \frac{\partial H}{\partial r}\frac{\partial p_\theta}{\partial p_r} - \frac{\partial H}{\partial p_r}\frac{\partial p_\theta}{\partial p_r} + \frac{\partial H}{\partial \theta}\frac{\partial p_\theta}{\partial p_\theta} - \frac{\partial H}{\partial p_\theta}\frac{\partial p_\theta}{\partial \theta} =$$

$[H, p_\theta] = 0$

(all terms above are 0)

and if we calculate

$[H, H] = 0$ and $[p_\theta, p_\theta] = 0$

Our system is then integrable

Let's find $r(t)$, $p_r(t)$, $\theta(t)$ and $p_\theta(t)$

Hamilton eqs are

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}} \qquad \boxed{\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \\ p_\theta &= \text{constant} \end{aligned}}$$

then $\frac{d\theta}{dt} = \frac{p_\theta}{mr^2} \rightarrow \int d\theta = \int \frac{p_\theta}{mr^2} dt$

and $\dot{r} = \frac{\partial H}{\partial p_r}$ $\qquad \dot{p}_r = -\frac{\partial H}{\partial r}$

$\rightarrow \quad \dot{r} = \dfrac{p_r}{m}$ good

$$\dot{p}_r = + \frac{p_\theta^2}{m r^3} - k r$$

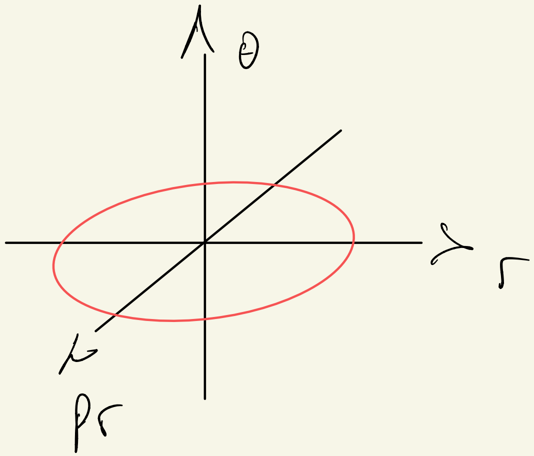
From $H \rightarrow$

$$p_r = \sqrt{2m\left(E - \frac{p_\theta^2}{2mr^2} - \frac{1}{2} k r^2\right)}$$

$$\boxed{\int \frac{m r \, dr}{\sqrt{2Emr^2 - p_\theta^2 - kmr^4}} = \int dt = t}$$

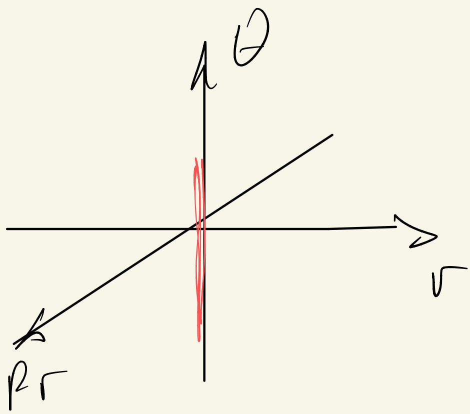
$$\boxed{\int dp_r = p_r = \int \left(\frac{p_\theta^2}{mr^3} - k r \right) dt}$$

A look into phase space



a) trajectory with $p_\theta = 0$

θ = constant.

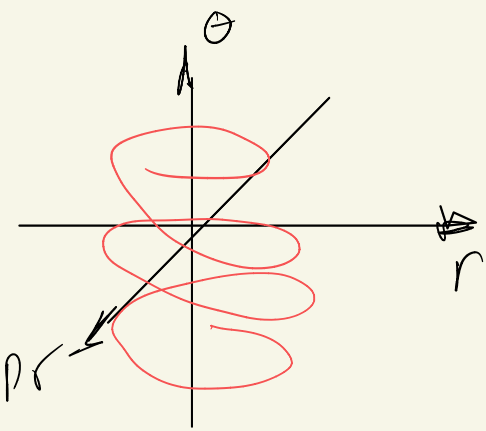


b) trajectory with $\dot{\theta} = \sqrt{\frac{k}{m}}$

$\ddot{\theta} = 0$!

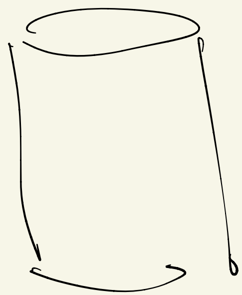
(r = constant)

then $p_r = 0$



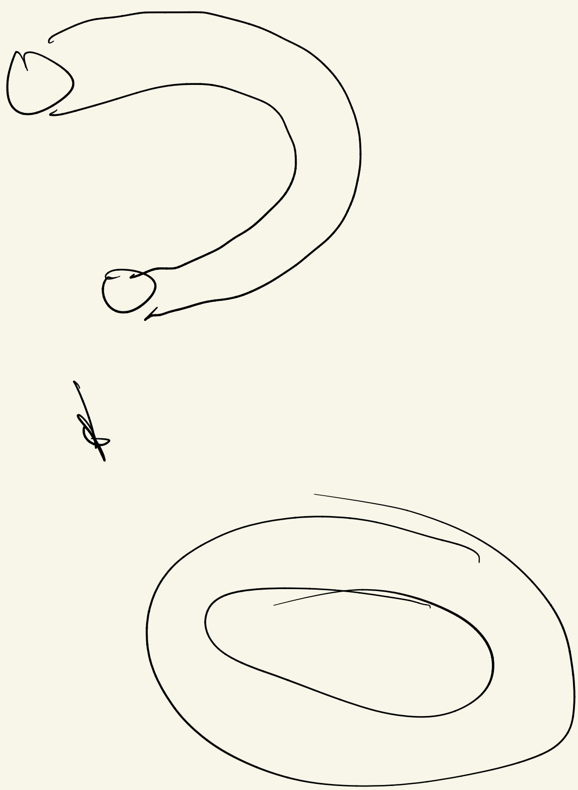
c) A generic trajectory

Notice that motion occurs on the surface of a cylinder.



let connect

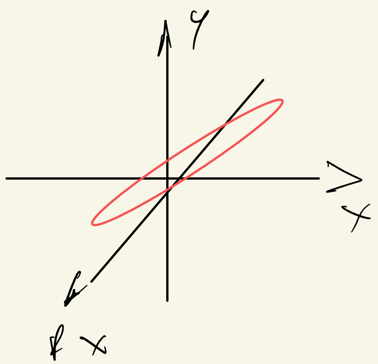
$\theta = 0 \rightarrow \theta = 2\pi$

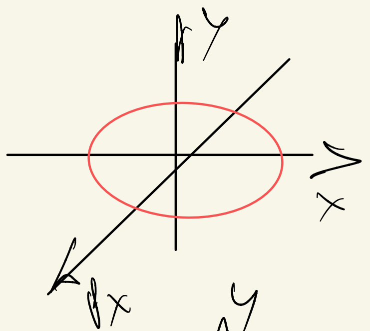


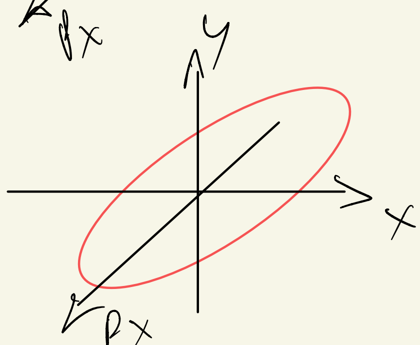
torus

Let's look at the same problem in Cartesian coordinates.

$$H = \frac{p_x^2}{2m} + \frac{1}{2} k x^2 + \frac{p_y^2}{2m} + \frac{1}{2} k y^2$$







We can imagine now

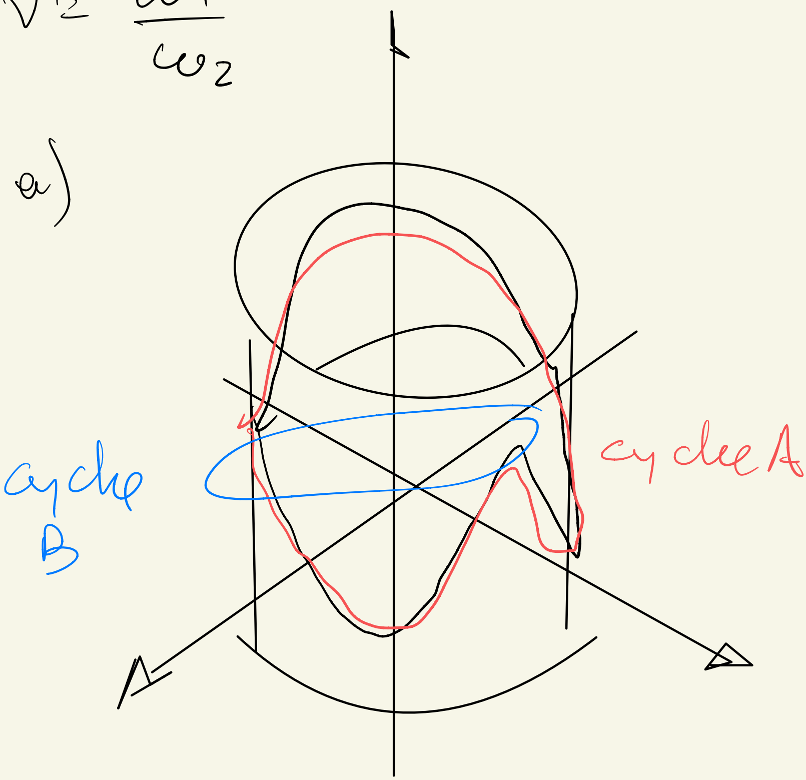
$$H = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_1^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2} m \omega_2^2 y^2$$

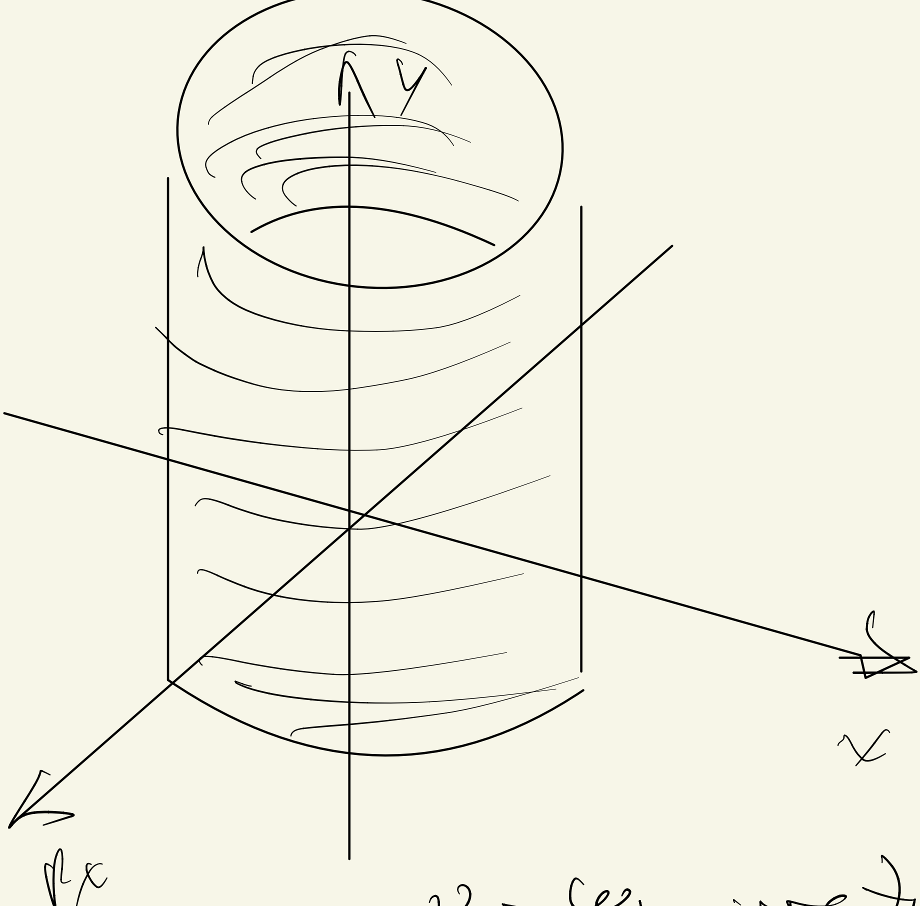
with $\omega_1 \neq \omega_2$

The energy of each oscillator (now there are 2!) is conserved and they are in involution.

$$\nu = \frac{\omega_1}{\omega_2}$$

a)





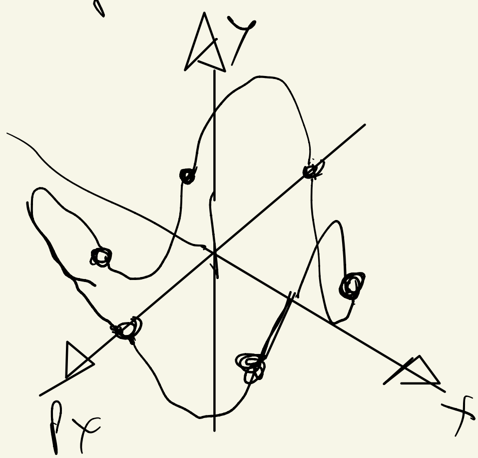
$\nu = \frac{\omega_1}{\omega_2}$ irrational

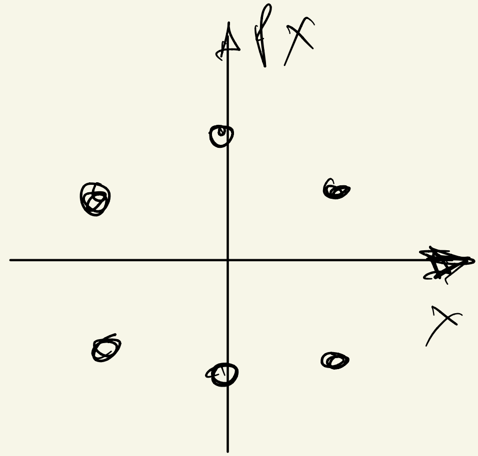
The intersection of the N-subspaces defined by $C_i = C_i(q,p)$ is a torus in a $2N-N = N$ dimensional space. It can be identified as a hypercube with opposing ends identified.

For an integrable system in 2-N phase space, the trajectory is confined to the surface of an N dimensional torus.

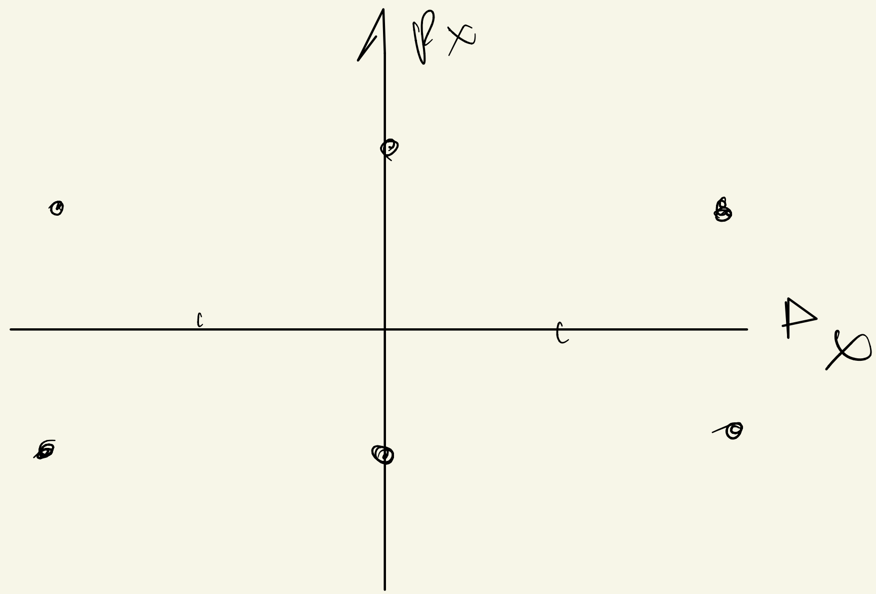
Non integrable systems

Let's look at 2-D snap shots. going back to the 2-D oscillator

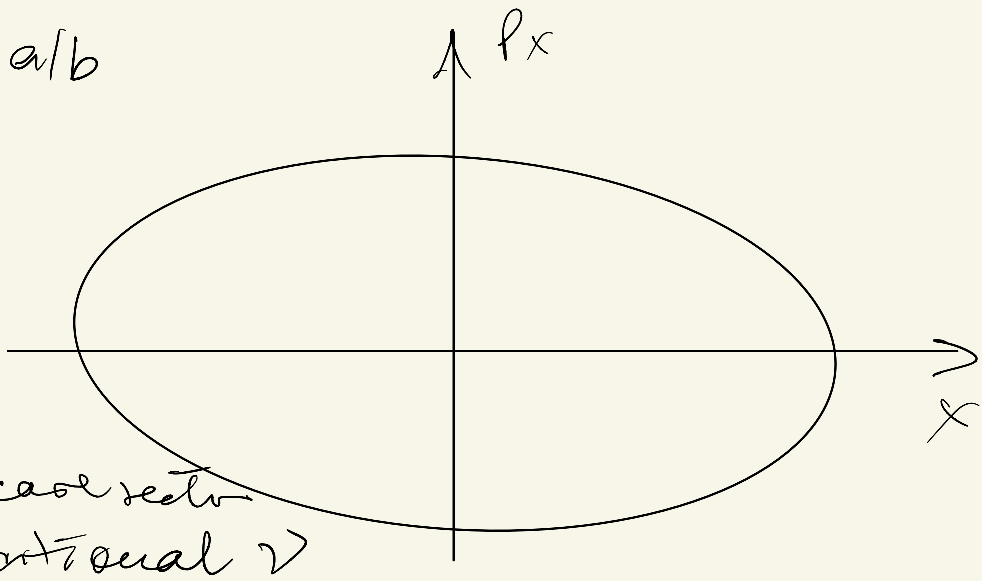




We can ~~read~~ ν from the number of times it circles the central point

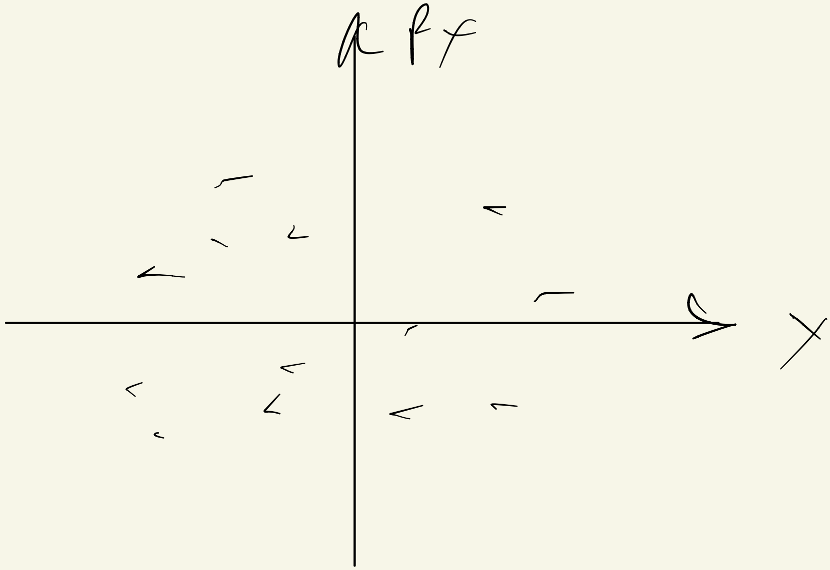


Poincare section for a rational $\nu = a/b$

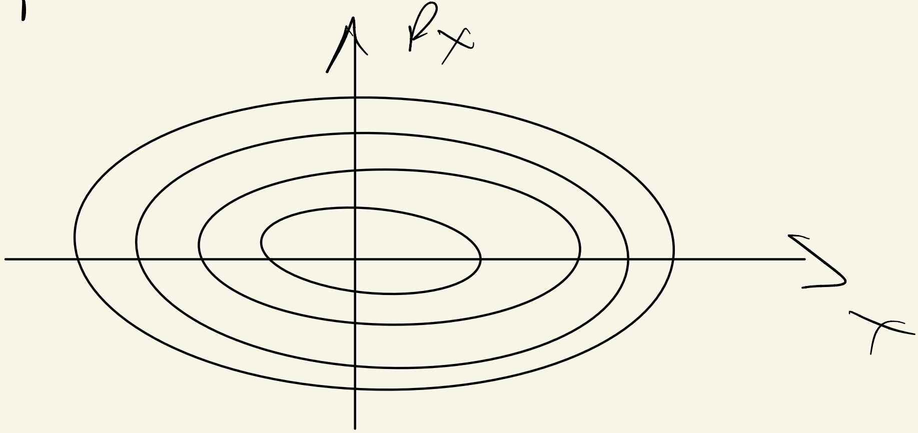


Poincare section irrational ν

These curves are called
KAM tori
(Kolmogorov, Arnol'd and Moser)
The 2 dimensional maps are
called Poincare sections
(or maps)
For integrable systems
Poincare sections consist
of a collection of closed loops
and discrete points when
we go over different initial
conditions.



Poincare section w/ rational winding numbers for various initial conditions responding to different total energies.



Consider a setup w/ 2 equal frequency harmonic oscillators coupled to each other.

$$H = \frac{p_1^2}{2m} + \frac{1}{2} m \Omega^2 q_1^2 + \frac{p_2^2}{2m} + \frac{1}{2} m \Omega^2 q_2^2$$

$$+ \frac{1}{2m^{3/2}} q_1 q_2^2 - \frac{1}{6m^{3/2}} q_1^3$$

This is the Henon-Heiles system. (2-D potential)

It was thought to model the dynamics of a star in the axisymmetric potential of a galaxy (q_1 the radial location of the star from the center of the galaxy and q_2 the angle of declination).

If we flip the sign for the last term.

$$H = \frac{p_1^2}{2m} + \frac{1}{2} m \Omega^2 q_1^2 + \frac{p_2^2}{2m} + \frac{1}{2} m \Omega^2 q_2^2$$
$$+ \frac{1}{2m^{3/2}} q_1 q_2^2 + \frac{1}{6m^{3/2}} q_1^3$$

this system is integrable
(not obvious)
If our goal is to analyse what happens as an integrable system turns into a chaotic one.
One technique → Henon-Heiles
→ non linear term right away
Another technique is to introduce the perturbation gradually
Conservative chaos: the Hamiltonian is independent of time.
If we add dissipation & time dissipative chaos?

Conservative chaos

Same system as before but with rest length of spring $\underline{r_0}$ (instead of 0) and with gravity

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} k (r - r_0)^2 + m g r \cos\theta$$

Energy

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} k (r - r_0)^2 - m g r \cos\theta$$

$\underline{E}$ constant

Angular momentum no longer conserved

We have 1 conserved quantity now

for a 2 DOF system.

Let's solve the equations

numerically

A natural scale is given by r_0

$$R = \frac{r}{r_0}$$

also we can define $T = \sqrt{\frac{k}{m}}\, t$

T is the # of oscillations periods

So $d/dt \to \frac{d}{dt} = \sqrt{\frac{k}{m}}\,\frac{d}{dT}$

$L \Rightarrow$ in terms of R and T

$$L = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2\right) - \frac{1}{2}(R-1)^2$$

$$+ \epsilon R \cos\theta$$

and $\epsilon \equiv \frac{mg}{k r_0}$

Let's start with $\epsilon = 0$

Assume $\varepsilon << 1$

R near eq. is $R_{eq} = 1 + \epsilon$

$$R = R_{eq} + x = 1 + \varepsilon + x$$

$$\cos\theta \simeq 1 - \frac{\theta^2}{2}$$

$$x << R_{eq} \qquad \theta << 1$$

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{2}(1+\epsilon)^2\dot{\theta}^2$$
$$- \frac{1}{2}\varepsilon(1+\varepsilon)\theta^2$$

This is equivalent to 2 harmonic oscillators & 2 angular frequencies

$$\omega_x = 1 \qquad \omega_\theta = \sqrt{\frac{\varepsilon}{1+\varepsilon}}$$

With this Poincare section would be KAM tori

The winding number is

$$\nu = \frac{\omega_\theta}{\omega_x} = \sqrt{\frac{\varepsilon}{1+\varepsilon}}$$

in general irrational (except for special values of ε)

small $x, \theta \Rightarrow x$ & θ stay small

If we look beyond this for small but fixed perturbation ε, the winding #s of KAM tori for ~~different~~ initial conditions become functions of the initial conditions as in figure (see photo)

At the center is a single point KAM torus of $\nu = 1$

For $\theta - p_\theta$ this corresponds to stable vertical oscillations at fixed $\theta = 0$

These points on Poincaré sections are stable tori (fixed points or elliptic points)

KAM theorem

The theorem is concerned with the stability of motions in hamiltonian systems, that are small perturbations of integrable hamiltonian systems.

If a small perturbation is added to an integrable system, initial conditions corresponding to irrational KAM tori in Poincaré sections are affected by minimal deformations with dynamics non chaotic

However tori with rational winding numbers are expected to disintegrate into chaotic motion.

Are irrational #s more or less irrational?

Dirichlet's theorem

(Approximation theorem or Diophantine approximation)

For any real numbers α and N with $1 \le N$ there are integers p and q such $1 \le q \le N$ and

$$|q\alpha - p| \le \frac{1}{\lfloor N \rfloor + 1} < \frac{1}{N}$$

where $\lfloor N \rfloor$ is the integer part of N!

For example

$$\nu = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 + \cdots}}}$$

c_is are unique integers.

The more c_is we need to approximate ν (for a required precision) the "less rational" is ν.

The most "irrational" number corresponds to the minimum value for the c_is, i.e $c_i = 1$ which is

$$\nu = \sqrt{5} - 1 \big/ 2 \simeq 0.618033988$$

which is the inverse of the golden ratio.

Intuitively rational winding #s are a sequence of disconnected dots that either form scatter around and destroy the Tori

irrational #s on the other hand. diffuse the effect of the perturbations leading to stable Tori

Most initial conditions correspond to stable KAM tori (almost #s between 0 and 1 are irrational) we can define chaos as the absence of integrability

But for different initial conditions, we may encounter regions of stability - particularly when the non-integrable nonlinearities are small.

regions of chaos and regions of ~~K~~AM stability.

Quantifiable new definition of chaos.

i.e. we shift a given set of initial conditions by a tiny amount and compute the distance d between 2 trajectories

$$d^2(t) = \sum_i (q_i'(t) - q_i(t))^2$$

q to shifted q'

If we found that at long times we have

$$d(t) \propto e^{\lambda t}$$

with $\lambda > 0$ we say that the corresponding initial conditions lead to chaotic evolution.

λ Lyapunov exponent

then regions of chaos in a Poincare section can be labeled by Lyapunov exponents

if instead $\lambda \leq 0$ attractor behavior dynamics attracted to a stable torus in phase space.

Ergodicity

Evolution of a system in phase space is said to be ergodic if given enough time, the trajectory of the system comes arbitrarily close to any point in phase space.

chaos $\rightarrow$ implies $\rightarrow$ ergodicity.