

Lesson 8:

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October 22, 2026

Hamilton's equations

Sometimes it is more convenient to use generalized coordinates and generalized momenta instead of generalized coordinates and velocities like we were doing so far: $q_i, \dot{q}_i \rightarrow q_i, p_i$.

We assume that i takes values ranging from 1 to n the number of degrees of freedom corresponding to the physical system under consideration.

The transformation of coordinates that it is required can be done applying a *Legendre transformation*.

Starting with the the Lagrangian we can assume that a small change in the generalized coordinates dq_i and the generalized velocities $d\dot{q}_i$ will induce a differential transformation in the Lagrangian function which to first order is:

$$\begin{aligned} dL &= \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \\ &= \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i \end{aligned} \quad (1)$$

where the last equation is the result of substituting $\frac{\partial L}{\partial \dot{q}_i} = p_i$ and $\dot{p}_i = \frac{\partial L}{\partial q_i}$ using Lagrange equations.

But we also notice that $\sum_i p_i d\dot{q}_i$ can be written as a total differential:

$$\sum_i p_i d\dot{q}_i = d \left(\sum_i p_i \dot{q}_i \right) - \sum_i \dot{q}_i dp_i \quad (2)$$

Using this result in (1)

$$dL = \sum_i \dot{p}_i dq_i + d \left(\sum_i p_i \dot{q}_i \right) - \sum_i \dot{q}_i dp_i \quad (3)$$

from which we obtain

$$d \left(\sum_i p_i \dot{q}_i - L \right) = - \sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \quad (4)$$

or defining a new function which it is called the Hamiltonian*:

$$H(p, q, t) = \sum_i p_i \dot{q}_i - L \quad (5)$$

which is precisely what we define as the energy of the system in the Lagrangian formalism. From this definition (4) reads:

$$dH = - \sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \quad (6)$$

*Sir William Rowan Hamilton (4 August 1805 – 2 September 1865) was an Irish mathematician, physicist and astronomer.

From where it is simple to read the Hamilton equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (7)$$

Notice that these are a set of $2n$ first order differential equations replacing the n set of second order differential equations which are the Euler-Lagrange equations. These equations of motion in the Hamiltonian formalism are also called the “canonical equations”.

Conservation of Energy

The total time derivative of the Hamiltonian is:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial H}{\partial p_i} \dot{p}_i = \frac{\partial H}{\partial t} \quad (8)$$

The last equation is because, due to Hamilton equations, $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ and then the second term in (8) is equal to the third one.

We can see then that if the Hamiltonian doesn't depend explicitly on time,

$$\frac{dH}{dt} = 0 \quad (9)$$

and the energy is conserved.

Like the Lagrangian, the Hamiltonian involve parameters. These parameters characterize the physical system itself. Let λ be such a parameter. Then

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i + \left(\frac{\partial L}{\partial \lambda} \right) d\lambda \quad (10)$$

on the other side the Hamiltonian

$$dH = - \sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \left(\frac{\partial L}{\partial \lambda} \right) d\lambda \quad (11)$$

which implies that

$$\left(\frac{\partial H}{\partial \lambda} \right) \bigg|_{p,q} = - \left(\frac{\partial L}{\partial \lambda} \right) \bigg|_{q,\dot{q}} \quad (12)$$

where p, q and $q, \dot{q}$ remain constant in each case.

We can arrive at the same conclusion in a different manner.

Let the Lagrangian for a given physical system be of the form $L = L_0 + L'$ where L' is a small correction to L_0 .

Then we can define a new Hamiltonian for our system H' as in,

$$H = H_0 + H' \quad (13)$$

such that

$$H' \Big|_{p,q} = -L' \Big|_{q,\dot{q}} \quad (14)$$

and

$$\left(\frac{\partial H}{\partial t}\right)\bigg|_{p,q} = -\left(\frac{\partial L}{\partial t}\right)\bigg|_{q,\dot{q}} \quad (15)$$

The Routhian

Let's suppose there are generalized coordinates and velocities $\rho, \xi, \dot{\rho}, \dot{\xi}$ and we want to transform them to $\rho, \xi, p_\rho, \dot{\xi}$ where p_ρ is the conjugate momentum to ρ . In that case we will have

$$\begin{aligned} dL &= \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial \xi} d\xi + \frac{\partial L}{\partial \dot{\xi}} d\dot{\xi} \\ &= \dot{p} dq + p d\dot{q} + \frac{\partial L}{\partial \xi} d\xi + \frac{\partial L}{\partial \dot{\xi}} d\dot{\xi} \end{aligned} \quad (16)$$

From where we can rearrange

$$d(L - p\dot{q}) = \dot{p}dq + p d\dot{q} + \frac{\partial L}{\partial \xi} d\xi + \frac{\partial L}{\partial \dot{\xi}} d\dot{\xi} \quad (17)$$

And we can define the Routhian as the following function

$$R(q, p, \xi, \dot{\xi}) = p\dot{q} - L \quad (18)$$

from where we obtain

$$dR = -\dot{p}dq + \dot{q}dp - \frac{\partial L}{\partial \xi} d\xi - \frac{\partial L}{\partial \dot{\xi}} d\dot{\xi} \quad (19)$$

and the following equations of motion:

$$\begin{aligned}\dot{q} &= \frac{\partial R}{\partial p}, & \dot{p} &= -\frac{\partial R}{\partial q} \\ \frac{\partial L}{\partial \xi} &= -\frac{\partial R}{\partial \xi}, & \frac{\partial L}{\partial \dot{\xi}} &= -\frac{\partial R}{\partial \dot{\xi}}\end{aligned}\quad (20)$$

with the addition of the second order differential equation

$$\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{\xi}} \right) = \frac{\partial R}{\partial \xi} \quad (21)$$

In conclusion the Routhian is a Hamiltonian for the ρ coordinate and a Lagrangian for ξ .

To write an expression for the energy of the system in terms of the Routhian

$$E = \dot{q} \frac{\partial L}{\partial \dot{q}} + \dot{\xi} \frac{\partial L}{\partial \dot{\xi}} - L = p\dot{q} + \dot{\xi} \frac{\partial L}{\partial \dot{\xi}} - L \quad (22)$$

which gives explicitly in terms of R

$$E = R - \dot{\xi} \frac{\partial R}{\partial \dot{\xi}} \quad (23)$$

Why bother defining a function such that the Routhian?

When a coordinate is cyclic it might be a convenient way of finding the equations of motion.

For example, if ρ is a cyclic coordinate the Routhian depends only on $p, \xi, \dot{\xi}$. Using (20) we immediately find that

$$\dot{p} = -\frac{\partial R}{\partial q} = 0 \quad (24)$$

from where we can quickly conclude that p is a constant. Also in this case

$$\frac{d}{dt} \left(\frac{\partial R(p, \xi, \dot{\xi})}{\partial \dot{\xi}} \right) = \frac{\partial R(p, \xi, \dot{\xi})}{\partial \xi} \quad (25)$$

where it is clear that only ξ is involved.

Once ξ is obtained we can solve

$$\dot{q} = \frac{\partial R}{\partial p} \quad (26)$$

by direct integration.

Poisson Brackets

Let $f(p, q, t)$ be some function of the coordinates, momentum and time, Its total time derivative is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial p_k} \dot{p}_k \right) \quad (27)$$

$\dot{q}_k$ and $\dot{p}_k$ are given by the canonical equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$.

And replacing above we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (28)$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f] \quad (29)$$

where

$$[H, f] = \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (30)$$

$[H, f]$ is called the Poisson bracket of h and f .

If the quantity f is an integral of motion $df/dt = 0$ and then in (29) we obtain

$$\frac{\partial f}{\partial t} + [H, f] = 0 \quad (31)$$

If the function f is additionally independent explicitly of time,

$$[H, f] = 0 \quad (32)$$

The Poisson bracket of the Hamiltonian and the function which is an integral of motion is zero.

This definition can be extended to two functions

$$[f, g] = \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right) \quad (33)$$

Properties of the Poisson bracket

Following the definition it is easy to prove the following properties

$$[f, g] = -[g, f] \quad (34)$$

$$[f, c] = 0 \quad (35)$$

if c is a constant. Also if we have two functions f_1, f_2 ,

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g] \quad (36)$$

$$[f_1 \times f_2, g] = f_1[f_2, g] + f_2[f_1, g] \quad (37)$$

If we want to take the time derivative of the Poisson bracket (33)

$$\frac{\partial}{\partial t} [f, g] = \left[\frac{\partial f}{\partial t}, g \right] + \left[f, \frac{\partial g}{\partial t} \right] \quad (38)$$

If one of the functions f and g is one of the momenta or coordinates the Poisson bracket reduces to a partial derivative

$$[f, q_k] = \frac{\partial f}{\partial p_k} \quad (39)$$

$$[f, p_k] = -\frac{\partial f}{\partial q_k} \quad (40)$$

From where we can obtain

$$[q_i, q_k] = 0 \quad [p_i, p_k] = 0 \quad [p_i, q_k] = \delta_{ik} \quad (41)$$

If we have three functions

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \quad (42)$$

This is called the Jacobi's identity.

It can be proved by direct calculation (brute force).
But it's quite important in defining the nature of the
Poisson brackets.

Poisson's theorem

If f and g are integrals of motion, their Poisson bracket is also an integral of motion,

$$[f, g] = \text{constant} \quad (43)$$

Proof

Let's apply the Jacobi's identity (42) using as a function h , the Hamiltonian of the system itself, ie. $h = H$.

$$[H, [f, g]] + [f, [g, H]] + [g, [H, f]] = 0 \quad (44)$$

but due to the fact that f and g are integrals of motion

$$[H, g] = 0 = [H, f] \quad (45)$$

So in (53) the only surviving term is

$$[H, [f, g]] = 0 \quad (46)$$

which indicates that $[f, g]$ is indeed an integral of motion as well.

An alternative and instructive way of also proving this result is by directly calculating the time deriva-

tive of the Poisson bracket using (29):

$$\frac{d}{dt}[f, g] = \frac{\partial}{\partial t}[f, g] + [H, [f, g]] \quad (47)$$

and using (38) we get

$$\begin{aligned} \frac{d}{dt}[f, g] &= \left[\frac{\partial f}{\partial t}, g \right] + \left[f, \frac{\partial g}{\partial t} \right] - [f, [g, H]] - [g, [H, f]] \\ &= \left[\frac{\partial f}{\partial t} + [H, f], g \right] + \left[f, \frac{\partial g}{\partial t} + [H, g] \right] \\ &= \left[\frac{df}{dt}, g \right] + \left[f, \frac{dg}{dt} \right] \end{aligned} \quad (48)$$

and clearly if f and g are integrals of motion it proves the theorem.

The action as a function of the coordinates

Let's consider the action as a function of the upper limit of integration t_2 , i.e.

$$S = \int_{t_1}^{t_2} L dt \quad (49)$$

Looking at a variation of the action

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt \quad (50)$$

The second term in the above equation, the integral, is zero, because of Lagrange's equations.

For the first term in the sum we put $\delta q(t_1) = 0$ and $\delta q(t_2) = \delta q$. Then (50) becomes

$$\delta S = p\delta q \quad (51)$$

or assuming n degrees of freedom we get

$$\delta S = \sum_i p_i \delta q_i \quad (52)$$

which implies

$$\frac{\partial S}{\partial q_i} = p_i \quad (53)$$

On the other hand from the definition of action

$$\frac{dS}{dt} = L \quad (54)$$

which leads to

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}_i \quad (55)$$

From where we have

$$\frac{\partial S}{\partial t} = \frac{dS}{dt} - \sum_i p_i \dot{q}_i = L - \sum_i p_i \dot{q}_i = -H \quad (56)$$

From here we can write

$$dS = \sum_i p_i dq_i - H dt \quad (57)$$

If δS is caused by a change in q_i and t at the beginning and end of the path

$$dS = \sum_i p_i^{(2)} dq_i^{(2)} - H^{(2)} dt^{(2)} - \sum_i p_i^{(1)} dq_i^{(1)} - H^{(1)} dt^{(1)} \quad (58)$$

The only possible motion is the one for which dS is a perfect differential (i.e. $dS = 0$).

The principle of action imposes restrictions on the range of possible motions.

Interesting enough the Hamilton's equations can be derived from defining the action

$$S = \int \left(\sum_i p_i dq_i - H dt \right) \quad (59)$$

we then have

$$\delta S = \int \left(\delta p dq + p d\delta q - \frac{\partial H}{\partial q} \delta q dt - \frac{\partial H}{\partial p} \delta p dt \right) \quad (60)$$

Integrating by parts the second term we get

$$\delta S = \int \delta p \left(dq - \frac{\partial H}{\partial p} dt \right) + [p, \delta q] - \int \delta q \left(dp + \frac{\partial H}{\partial q} \right) dt \quad (61)$$

at the limits of integration we have $\delta q = 0$.

The two integrands will have to vanish because δp and δq vary independently and arbitrarily. Then

$$dq = \frac{\partial H}{\partial p} dt \quad dp = -\frac{\partial H}{\partial q} dt \quad (62)$$

and dividing by dt are the Hamilton's equations.

Maupertuis' Principle

There is a simplified form of the Principle of least action which is useful if we are more interested in finding the path to be followed by a physical system than the precise position as a function of time.

Let's assume that the energy is conserved, and consequently both the Hamiltonian and the Lagrangian do not depend explicitly of time.

$$H(p, q) = E = \text{constant} \quad (63)$$

We will also allow for a variation of the final time we are considering for a given path (instead of t_0, t we will consider $t_0, t + \delta t$). We have

$$\delta S = -H\delta t \quad (64)$$

We are comparing not all possible virtual motions but only those which conserve the energy,

$$\delta S + E\delta t = 0 \quad (65)$$

Writing the action as

$$S = \int \left(\sum_i p_i dq_i - H dt \right) \quad (66)$$

From where we obtain

$$S = \int \sum_i p_i dq_i - E(t - t_0) \quad (67)$$

The first term above is called the abbreviated action

$$S_0 = \int \sum_i p_i dq_i \quad (68)$$

The abbreviated action fulfills (67 in 66)

$$\delta S_0 = 0 \quad (69)$$

It has a minimum with respect to all paths which satisfy the law of conservation of energy and pass through the final point at any instant.

In order to use this principle, the momenta need to be expressed as a functions of q and dq .

$$p_i = \frac{\partial}{\partial \dot{q}_i} L(q, \dot{q}) \quad (70)$$

and

$$E = E(q, \dot{q}) \quad (71)$$

Expressing dt in terms of q_i and dq_i in (71) and substituting in (70) we will have the momenta in terms of q_i and dq_i with E as a parameter.

The variational principle obtained in this manner is called the Maupertuis' Principle.

In the case that the Lagrangian takes its original form

$$L = \frac{1}{2} \sum_{ik} a_{ik}(q) \dot{q}_i \dot{q}_k - U(q) \quad (72)$$

where the momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_k a_{ik}(q) \dot{q}_k \quad (73)$$

and

$$E = \frac{1}{2} \sum_{ik} a_{ik}(q) \dot{q}_i \dot{q}_k + U(q) \quad (74)$$

where the last equation gives

$$dt = \sqrt{\frac{\sum_{ik} a_{ik}(q) dq_i dq_k}{2(E - U)}} \quad (75)$$

In order to obtain an expression where we use E as a parameter we substitute in $\sum_i p_i dq_i = \sum_{ik} a_{ik} \frac{dq_k}{dt} dq_i$ and then in (68)

$$S_0 = \int \sqrt{2(E - U) \sum_{ik} a_{ik}(q) dq_i dq_k} \quad (76)$$

In the case of a single particle we have

$$T = \frac{1}{2} m \left(\frac{dl}{dt} \right)^2 \quad (77)$$

from where

$$S_0 = \int \sqrt{2(E - U)m} dl^2 \quad (78)$$

then Maupertuis

$$\delta S_0 = \delta \int \sqrt{2(E - U)m} dl = 0 \quad (79)$$

and if the particle is free $U = 0$ and

$$\sqrt{2Em} \int dl = 0 \quad (80)$$

which shows that the straight line is the shortest path in the absence of U . Returning to (67)

$$S = \int \sum_i p_i dq_i - E(t - t_0) \quad (67)$$

we will consider a variation respect to E

$$\delta S = \frac{\partial S_0}{\partial E} \delta E - (t - t_0) \delta E - E \delta t \quad (81)$$

Substituting in (65)

$$\delta S + E \delta t = 0 \quad (65)$$

we get

$$\frac{\partial S_0}{\partial E} = t - t_0 \quad (82)$$

where $S_0 = \int \sqrt{2(E - U) \sum_{ik} a_{ik}(q) dq_i dq_k}$ and

taking the derivative we get

$$\int \sqrt{\frac{\sum_{ik} a_{ik}(q) dq_i dq_k}{2(E - U)}} = t - t_0 \quad (83)$$

which is the integral of (84)

$$dt = \sqrt{\frac{\sum_{ik} a_{ik}(q) dq_i dq_k}{2(E - U)}}$$

Together with the equation for the path it determines the motion.

Canonical Transformations

In the Lagrangian formalism we can have s generalized coordinates $q_i \rightarrow Q_k$ such that $Q_i = Q_i(q, t)$.

In the Hamiltonian formalism we have $2s$ generalized coordinates which can be transformed

$$q, p \rightarrow Q_i = Q_i(p, q, t), P_i = P_i(p, q, t). \quad (84)$$

But the equations do not retain their canonical form under all possible transformations.

What are the conditions that need to be satisfied so that with new coordinates P, Q

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \quad P_i = -\frac{\partial H'}{\partial \dot{Q}_i} \quad (85)$$

for some H' .

One class of transformations is particularly important. These are called canonical transformations.

We know that Hamilton's equations can be obtained from

$$\delta \int \left(\sum_i p_i dq_i - H dt \right) = 0 \quad (86)$$

If the new coordinates P, Q satisfy the Hamilton's Principle

$$\delta \int (\sum_i P_i dQ_i - H' dt) = 0 \quad (87)$$

Then the difference between (85) and (86) should just be a function of coordinates and time

$$\sum p_i dq_i - H dt = \sum P_i dQ_i - H' dt + dF \quad (88)$$

which we can write as

$$dF = \sum p_i dq_i - \sum P_i dQ_i + (H' - H) dt \quad (89)$$

from where we obtain

$$p_i = \frac{\partial F}{\partial q_i}, \quad P_i = \frac{\partial F}{\partial Q_i}, \quad H' = H + \frac{\partial F}{\partial t} \quad (90)$$

F is called the generating function.

We assume $F = F(q, Q, t)$. When F is known equation (90) give the relationship between p, q and P, Q as well as the new Hamiltonian.

In some situations it may be more convenient to give F in terms not of q, Q but of q, P .

In these cases we apply a Legendre transformation

$$d(F + \sum P_i Q_i) = \sum_i p_i dq_i + \sum Q_i dP_i + (H' - H)dt \quad (91)$$

If we call $F + \sum P_i Q_i = \Phi(q, P, t)$ we have

$$p_i = \frac{\partial \Phi}{\partial q_i}, \quad Q_i = \frac{\partial \Phi}{\partial P_i}, \quad H' = H + \frac{\partial \Phi}{\partial t} \quad (92)$$

Similarly we can obtain canonical transformations involving generating functions of p, Q or p, P .

Always $H' - H$ is the time derivative of the generating function.

If f does not depend of the time $H' = H$.

Notice also that with this formalism we can have $Q_i = p_i$ and $P_i = -q_i$.

This is just an exchange of names for the variables. In this context of arbitrariness of the meaning of p and q , they are called canonically conjugated quantities.

Example

Consider the simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \quad (93)$$

where $\omega = \sqrt{\frac{k}{m}}$. Let's consider

$$F_1(q, Q, t) = qQ \quad (94)$$

From where using the relationships established in equation (92) we get

$$p_k = \frac{\partial F_1}{\partial q_k}, \quad P_k = \frac{\partial F_1}{\partial Q_k}, \quad H' = H + \frac{\partial F_1}{\partial t} \quad (95)$$

and consequently we find $p = Q$ and $P = -q$ and

$$H' = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + 0 = \frac{Q^2}{2m} + \frac{1}{2}m\omega^2 p^2 \quad (96)$$

and

$$\dot{Q} = \frac{\partial H'}{\partial P} = m\omega^2 P, \quad \text{and} \quad \dot{P} = \frac{\partial H'}{\partial Q} = -\frac{Q}{m} \quad (97)$$

which shows that Q, P is physically equivalent to q, p .

General Theorem about Poisson brackets

The conditions relating canonically conjugated variables can be expressed in terms of the Poisson brackets. If we have the Poisson brackets associated to two functions f and g calculated as derivatives respect to the coordinates p, q and P, Q then

$$[f, g]_{p,q} = [f, g]_{P,Q} \quad (98)$$

This can be demonstrated directly. But it can also be demonstrated in the following manner.

In (90) and (92) the time appears as a parameter.

This makes it sufficient to prove (98) for functions that do not depend explicitly on time.

Let's think that g is the Hamiltonian of some system. Then following (29)

$$[f, g]_{p,q} = -\frac{df}{dt} \quad (99)$$

In the equation above df/dt depends only of the physical system it describes, of course, and not on the variables utilized to describe it.

Then $[f, g]$ does not change when transforming from one set of canonical coordinates to the other.

Using together (41) and (98) P, Q need to satisfy

$$[Q_i, Q_k]_{p,q} = 0 \quad [P_i, P_k]_{p,q} = 0 \quad [P_i, Q_k]_{qp} = \delta_{ik} \quad (100)$$

if the transformation $q, p \rightarrow P, Q$ is canonical.

Notice that the time evolution of $p(t), q(t)$ can be thought of as a set of canonical transformations, where S the action is the generating function.