Estimating the accuracy in the conservation of the adiabatic invariant

The equation:

$$\dot{I} = -\frac{\partial H'}{\partial w} = -\left(\frac{\partial \Lambda}{\partial w}\right)_{I,\lambda} \dot{\lambda}, \quad (106)$$

is a further confirmation that I, the action function is an adiabatic invariant.

$S_0(q, I, \lambda)$ is a periodic function of q. Eventually when S_0 increases by $n\,2\pi I$ q returns to its original value.

$$\Lambda = \left(\frac{\partial S_0}{\partial \lambda}\right)_{q,I}$$

This derivative is single valued though.

This is because the derivative is at constant I and the changes in I do not appear.

Λ is a periodic function when expressed in terms of W

The mean value, over a period, of $\frac{\partial \Lambda}{\partial W}$, being periodic is zero.

So

$$\overline{\frac{dI}{dt}} = -\overline{\left(\frac{\partial \Lambda}{\partial W}\right)_I} \frac{\partial \Lambda}{\partial t} = 0$$

showing it again that I is an adiabatic invariant.

Looking at

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{\partial H'}{\partial W} = -\left(\dfrac{\partial \Lambda}{\partial W}\right)_{I,\lambda} \dfrac{d\lambda}{dt} \\[2ex] \dfrac{dW}{dt} = \dfrac{\partial H'}{\partial I} = \omega(I,\lambda) + \left(\dfrac{\partial \Lambda}{\partial I}\right)_{W,\lambda} \dfrac{d\lambda}{dt} \end{cases}$$

Remember $\omega = \left(\dfrac{\partial E}{\partial I}\right)_{\lambda}$ is the oscillation frequency.

Let consider λ_- and λ_+

where $\lambda_- = \lim_{t \to -\infty} \lambda(t)$

$\lambda_+ = \lim_{t \to +\infty} \lambda(t)$

I_- is the value of $I(\lambda_-)$

I_+ " " " " $I(\lambda_+)$

Then we can calculate ΔI

$$\Delta I = -\int_{-\infty}^{\infty} \frac{\partial \Lambda}{\partial W} \frac{d\lambda}{dt}\, dt$$

Λ is a periodic function of W with period 2π.
We can expand in a Fourier series

$$\Lambda = \sum_{\ell=-\infty}^{\infty} e^{i\ell W} \Lambda_\ell \qquad (107)$$

Λ is real $\longrightarrow$ $\Lambda_{-\ell} = A_\ell^*$

Then

$$\frac{\partial \Lambda}{\partial W} = \sum_{\ell=-\infty}^{\infty} i\ell\, e^{i\ell W} \Lambda_\ell$$

$$= 2\,\mathrm{Re} \sum_{\ell=1}^{\infty} i\ell\, e^{i\ell W} \Lambda_\ell \qquad (108)$$

when $\frac{d\lambda}{dt}$ is sufficiently small $\longrightarrow$

$\frac{dW}{dt} > 0$, i.e. W is a monotonic function of time

changing from $dt \to dW$

$$\Delta I = -\int_{-\infty}^{\infty} \frac{\partial \Lambda}{\partial W} \frac{d\lambda}{dt} \frac{dt}{dW} \, dW \quad (109)$$

Using $\frac{\partial \Lambda}{\partial W} = 2 \mathrm{Re} \sum_{l=1}^{\infty} i l e^{ilW} \Lambda_l$

in (109), treating W as a complex variable

Assume no real singularities and follow a path of integration in the complex plane

Refresher on Contour Integration

(line integrals in the complex plane)

Contour integration is a method of evaluating certain integrals along paths in the complex plane.

(Calculus of residues, Cauchy integral formula)

If $\quad f(t) = x(t) + i\,y(t)$

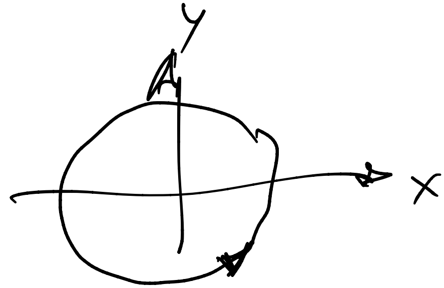
$$\int_a^b f(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

$f: \mathbb{C} \to \mathbb{C}$ on a curve γ

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\frac{d\gamma}{dt}\,dt$$

Example 1

$$\oint_C \frac{1}{z}\,dz$$



take $|z|=1 \rightarrow z(t)=e^{it}$

$t \in [0, 2\pi]$

$$\frac{dz}{dt} = ie^{it}$$

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\, ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Example 2

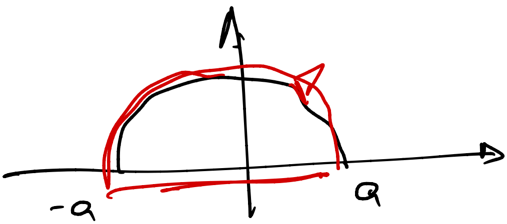
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$$

let's look at

$\rightarrow f(z) = \dfrac{1}{(z^2+1)^2}$

singularities at i and $-i$

Choose a contour that will enclose the real valued integral



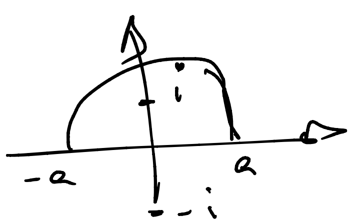
$$\oint_C f(z)\,dz = \int_{-a}^{a} f(z)\,dz + \int_{arc} f(z)\,dz$$

$$\int_{-a}^{a} f(z)\,dz = \oint_C f(z)\,dz - \int_{Arc} f(z)\,dz$$

Observe that

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$$

In the contour the only singularity is at i



$$f(z) = \frac{\frac{1}{(z+i)^2}}{(z-i)^2},$$

Using Cauchy method

$$\oint f(z)\,dz = \oint \frac{\frac{1}{(z+i)^2}}{(z-i)^2}\,dz = 2\pi i \left.\frac{d}{dz}\frac{1}{(z+i)^2}\right|_{z=i} =$$

$$= 2\pi i \left[\frac{-2}{(z+i)^3}\right]_{z=i} = \frac{\pi}{2}$$

we take the derivative because it is a second order pole.

On the other hand

$$\left|\int_{arc} f(z)\,dz\right| \leq ML$$

M upper bound of $f(z) = \frac{1}{(z^2+1)^2}$ along the arc and L the length of the arc →

$$L = a\pi$$

$$M = \frac{1}{(a^2-1)^2}$$

So $\left| \int_{arc} f(z) \right| \leq \frac{a\pi}{(a^2-1)^2} \xrightarrow[a \to \infty]{} 0$

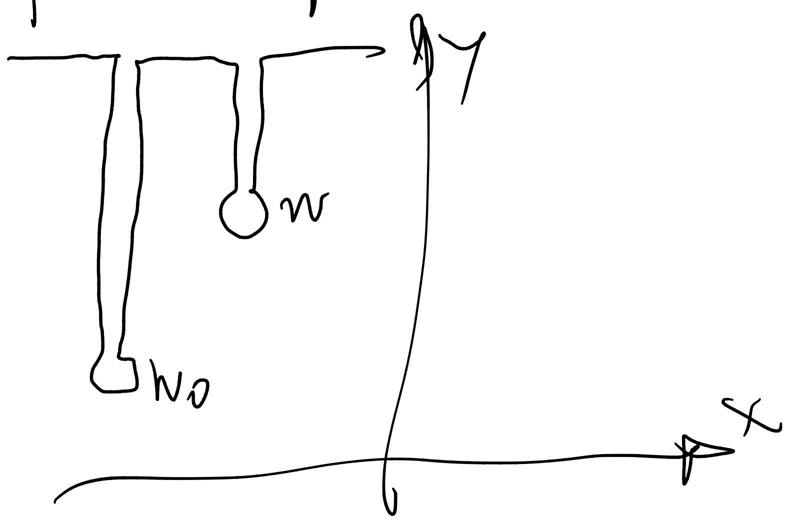
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx = \int_{-\infty}^{\infty} f(z)\,dz =$$

$$= \lim_{a \to \infty} \int_{a}^{-a} f(z)\,dz = \frac{\pi}{2}$$

back to

$$\Delta I = -\int_{-\infty}^{\infty} \frac{\partial \Lambda}{\partial W} \frac{dk}{dt} \frac{dt}{dW} dW \quad (110)$$

We can assume there are no singularities for real W and integrate off the real axis into half plane of the complex variable.
The contour could contain the singularities in the integrand we form loops around them



Let w_0 be the singularity closest to the real axis

Principal contribution to (110) comes from the neighborhood of this point.

Each term in (108)

$$\frac{\partial \Lambda}{\partial W} = 2\,\mathrm{Re} \sum_{l=1}^{\infty} i\, l\, e^{ilW} \Lambda_l \quad (111)$$

give a contribution containing a factor $\exp(-I\, i m W_0)$.

Retaining only the term with the negative exponent of smallest magnitude (lowest l)

$$\Delta I \sim \exp(-i m W_0) \quad (112)$$

$t_0 \to$ such $w(t_0) = W_0$

The order of magnitude of the exponent

$|t_0| \sim \tau$ $\qquad$ τ characteristic time of variation of parameters.

$\rightarrow \operatorname{im} W_0 \sim \omega\tau \sim \tau/T$

Since $\tau >> T$, exponent is large

ΔI decreases exponentially as the rate of variation of parameters decreases.

To determine W_0 in first approximation with respect to T/τ i.e. we keep only $\sim \frac{1}{(T/\tau)}$ in the exponent, we omit in

$$\frac{dW}{dt} = \frac{\partial H'}{\partial I} = \omega(I;\lambda) + \left(\frac{\partial \Lambda}{\partial I}\right)_{w,\lambda} \frac{d\lambda}{dt}$$

the term $\frac{d\lambda}{dt}$, ie

$$\frac{dW}{dt} = \omega\left(I, \lambda(t)\right)$$

I in $\omega(I, \lambda)$ has a fixed value

i.e $I\_$

Then

$$W_0 = \int^{t_0} \omega(I, \lambda(t))\, dt$$

Then

$$\Delta I = -\int_{-\infty}^{\infty} \frac{\partial \Lambda}{\partial W} \frac{d\lambda}{dt} \frac{dt}{dW} dW$$

becomes

$$\Delta I \sim \mathrm{Re} \int i e^{iW} \frac{\dot{\lambda}\, d}{\omega(I, \lambda)} \quad (113)$$

The singularities in question are the poles and branch points of $\dot{\lambda}(t)$ and $1/\omega(t)$

Reminder: We claim ΔI is exponentially small because we assume these functions have no real singularities.

Conditionally periodic motion

Let's consider a system with any # of degrees of freedom, executing a motion finite in all the coordinates and assume that we can separate variables as in H-J formalism.

So

$$S_0 = \sum_i^{n} S_i(q_i) \qquad (114)$$

also

$$p_i = \frac{\partial S_0}{\partial q_i} = \frac{d S_i}{d q_i} \qquad (115)$$

so

$$S_i = \int p_i \, dq_i \qquad (116)$$

If q_i cycles $\rightarrow$

$$\Delta S_0 = \Delta S_i = 2\pi I_i \qquad (117)$$

with $$I_i = \frac{\oint p_i \, dq_i}{2\pi} \qquad (118)$$

Let's do a canonical transformation $\left(W = \frac{\partial S_0}{\partial I} \right)$ now for many variables

$$\rightarrow \quad \frac{dI_i}{dt} = 0 \qquad \frac{dW_i}{dt} = \frac{\partial E(I)}{\partial I_i}$$

we get

$$I_i = \text{constant} \qquad (119)$$

$$W_i = \frac{\partial E(I)}{\partial I_i} t + \text{const} \qquad (120)$$

A cycle in q_i corresponds to

$$\Delta W_i = 2\pi \qquad (121)$$

Substitution of $I_i(p,q)$ in $W_i(q, I)$ gives $W_i(p,q)$ which may vary by $n\,2\pi$

Any function $F(p,q)$ of the state of the system, if expressed in terms of the canonical variables, is a periodic function of the angle variables and the period in each variable is 2π. So we can expand

$$F = \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_s=-\infty}^{\infty} A_{l_1 l_2 \ldots l_s} \exp\left[i\left(l_1 W_1 + \ldots + l_s W_s\right)\right]$$

where $l_1, \ldots, l_s$ are integers

$w_i \to \frac{\partial E}{\partial I_i}$ we get

$$F = \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_s=-\infty}^{\infty} A_{l_1 l_2 \ldots l_s} \exp\left\{ it\left(l_1 \frac{\partial E}{\partial I_1} + \cdots l_s \frac{\partial E}{\partial I_s} \right) \right\} \tag{122}$$

Each term in the sum is a periodic function of time with frequency

$$l\omega_1 + \cdots + l_s \omega_s \tag{123}$$

Which is a sum of integral multiples of the fundamental frequencies

$$\omega_i = \frac{\partial E}{\partial I_i} \tag{124}$$

The motion is not strictly periodic as a whole or in any coordinate. The system does not return to a given state in a finite period of time. But it passes arbitrarily closed.

→ <u>conditionally periodic</u>

In some cases 2 or more fundamental ω_i are commensurate for values of I_i → degeneracy. If all of them are is → complete degeneracy.

If 2 frequencies are, i.e. ω_1 & ω_2

$$n_1 \frac{\partial E}{\partial I_1} = n_2 \frac{\partial E}{\partial I_2} \qquad (125)$$

I_1 and I_2 appear as $n_2 I_1 + n_1 I_2$

An important property of degenerate motion is the increase in the number of one-valued integrals of motion over the number for an non-degenerate case

i.e out of $2s-1 \longrightarrow$ only s functions of state are 1-valued the s quantities I_i

The remaining $s-1$ integrals

$$w_i \frac{\partial E}{\partial I_k} - w_k \frac{\partial E}{\partial I_i} \qquad (126)$$

when that's the case although

$$w_1 n_2 - w_2 n_1$$

is not one valued; it is so

by the addition of an arbitrary integral multiple of 2π.

An example is $U = -\dfrac{\alpha}{r}$

there is an additional

$$\vec{v} \times \vec{M} + \alpha \frac{\vec{r}}{r}$$

besides $\vec{M}$ and E. (2-d problem)

<u>Note</u>: additionally degenerate motions allow a complete separation of variables for several choices of coordinates.

When degeneracy occurs the number of one-valued integrals exceeds <u>5</u> and there is no unique choice of I_i.

An example Keplerian motion → parabolic coordinates as well as spherical.

We already saw that the action variable I is an adiabatic invariant. True for more than 1-degree of freedom
For a multidimensional system with $N(t)$ EoM in canonical variables give

$$\frac{dI_i}{dt} = -\frac{\partial \Lambda}{\partial N_i} \frac{d\lambda_i}{dt} \qquad (127)$$

$$\Lambda = \frac{\partial S_0}{\partial N}$$

Then Λ is a unique function of N_i and the mean values of $\frac{\partial \Lambda}{\partial N_i} \to 0$

Example

Calculate the action variables for elliptic motion in $U = -\frac{\alpha}{r}$

Solution

ten r, ϕ

$$I_\phi = \frac{1}{2\pi}\int_0^{2\pi} p_\phi \, d\phi = M$$

$$I_r = 2\frac{1}{2\pi}\int_{r_{min}}^{r_{max}} \sqrt{2m\left(E+\frac{\alpha}{r}\right) - \frac{M^2}{r^2}}\, dr$$

$$= -M + \alpha\sqrt{\frac{m}{2|E|}}$$

The energy as a function of I is

$$E = \frac{-m\alpha^2}{2(I_r + I_\phi)^2}$$

degenerate (depends on the sum) the 2 fundamental frequencies in r and ϕ coincide $\left(\omega_i = \frac{\partial E}{\partial I_i}\right)$

Parameters p and e

$$p = \frac{M^2}{m\alpha} \qquad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$$

are

$$p = \frac{I_\phi^2}{m\alpha} \qquad e^2 = 1 - \left(\frac{I_\phi}{I_\phi + I_r}\right)^2$$

Since I_r I_ϕ are adiabatic invariants if α or m varies slowly e remains unchanged, its dimensions vary $1/\alpha$ & $1/m$.