Introduction to General Relativity 2026
Homework 1

Exercise 1.
Prove formula (10) from page 8 from the Lecture notes Lesson-1-2026.pdf.

Exercise 2.
How does the Lagrangian of a free particle:

$$L = -\sqrt{1 - \left(\frac{dx}{dt}\right)^2} \tag{1}$$

transforms under the following coordinate transformation (q, τ):

$$q = \cosh(\psi)x + \sinh(\psi)ct \tag{2}$$
$$\tau = \sinh(\psi)x + \cosh(\psi)ct \tag{3}$$

Discuss.

Exercise 3.
Prove formula (64) from page 17 from the Lecture notes Lesson-1-2026.pdf.

Exercise 4.
Use Noether's theorem for transformations that represent only a time translation, i.e. $t' = t + \epsilon t$, and see what quantity is conserved from

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \left(\dot{q}_i X - \Psi_i\right) - LX \tag{4}$$

under the following transformation from coordinates q_i, t to ${q_i}', t'$.

$${q_i}' = q_i + \epsilon \Psi_i(q, t) \tag{5}$$
$$t' = t + \epsilon X(q, t) \qquad \epsilon \to 0. \tag{6}$$

Repeat for $x' = t + \epsilon x$. What is the quantity conserved?

Exercise 5.

As we will see in next chapter the magnitude of a vector in a Minkowski space-time is a scalar of course which it is not necessarily a positive number. Vectors, In Minkowski S-T could have positive, negative or zero values. In general the magnitude of a vector is defined as

$$\vec{A} \cdot \vec{B} = A^{\alpha} B^{\beta} \eta_{\alpha\beta}, \tag{7}$$

The numbers $\eta_{\alpha\beta}$ are called the components of the metric tensor. In the case of Minkowski S-T

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

If for example the vector $\vec{a} = (A^0, A^1, A^2, A^3)$, its magnitude will be given by

$$\vec{A} \cdot \vec{A} = (A^0, A^1, A^2, A^3) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2 \tag{9}$$

For the pairs of events whose coordinates (t, x, y, z) in some frame are given below, classify their separations as timelike, spacelike, or null.

(a) (0, 0, 0, 0) and (-1,1, 0, 0),

(b) (1, 1, -1, 0) and (-1, 1, 0, 2),

(c) (6, 0, 1, 0) and (5, 0, 1, 0),

(d) (-1, 1, -1, 1) and (4, 1, -1, 6).

Exercise 6.

Show that the hyperbolae $-t^2 + x^2 = a^2$ and $-t^2 + x^2 = -b^2$ are asymptotic to the lines $t = \pm x$, regardless of a and b.

(Hint: To show a hyperbola is asymptotic to a line, substitute the line's equation into the hyperbola's equation and show the resulting quadratic equation has roots that tend to infinity).