Introduction to GR-2026
Homework 2

1 Exercise 1

Given the numbers:
$\{A^0 = 5, A^1 = 0, A^2 = -1, A^3 = -6\}, \{B_0 = 0, B_1 = -2, B_2 = 4, B_3 = 0\}$, and,
$C = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 5 & -2 & -2 & 0 \\ 4 & 5 & 2 & -2 \\ -1 & -1 & -3 & 0 \end{pmatrix}$ find:

(a) $A^\alpha B_\alpha$;
(b) $A^\alpha C_{\alpha\beta}$ for all β;
(c) $A^\gamma C_{\gamma\sigma}$ for all σ;
(d) $A^\nu C_{\mu\nu}$ for all μ;
(e) $A^\alpha B_\beta$ for all α, β;
(f) $A^i B_i$;
(g) $A^j B_k$ for all j, k;

2 Exercise 2

Identify the free and dummy indices in the following equations and change them into equivalent expressions with different indices. How many different equations does each expression represent?
(a) $A^\alpha B_\alpha = 5$;
(b) $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu} A^\nu$;
(c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha}$;
(d) $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu}$;

3 Exercise 3

A collection of vectors $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is said to be linearly independent if no linear combination of them is zero except the trivial one, $0\vec{a} + 0\vec{b} + 0\vec{c} + 0\vec{d} = 0$.
(a) Show that the basis vectors $\vec{e}_0 = (1, 0, 0, 0), \vec{e}_1 = (0, 1, 0, 0), \vec{e}_2 = (0, 0, 1, 0), \vec{e}_3 = (0, 0, 0, 1)$ are linearly independent.
(b) Is this set of basis vectors $\{\vec{a}, \vec{b}, \vec{c}, 5\vec{a} + 3\vec{b} - 2\vec{c}\}$ LI?

4 Exercise 4

(a) Prove that the zero vector $(0, 0, 0, 0)$ has these same components in all reference frames.
(b) Use (a) to prove that if two vectors have equal components in one frame, they have equal components in all frames.

5 Exercise 5

(a) Show that the sum of any two orthogonal space-like vectors is spacelike.
(b) Show that a time-like vector and a null vector cannot be orthogonal.

6 Exercise 6

Write down the change of coordinates from Cartesian (x, y, z) to spherical coordinates (r, θ, ϕ) Obtain the transformation matrices from one to the other. Write $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$ as vector operators in both coordinate systems. Show how the gradient in the direction orthogonal to spheres of constant radius transform from one system to the other.