INTRODUCTION TO GENERAL RELATIVITY AND GRAVITATION

HOMEWORK 8

Exercise 1.

Let's look at a particle that is in the background of a plane gravitational wave. The particle has velocity U^α such that the geodesic equation:

$$\frac{d}{d\tau}U^\alpha + \Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu = 0 \tag{1}$$

Show explicitly that before the wave arrives (the particle is at rest) the initial value of the acceleration is:

$$\frac{d}{d\tau}U^\alpha \mid_0 = -\Gamma^\alpha{}_{00} = -\frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}) \tag{2}$$

Exercise 2.

Show that in the TT gauge of linearized gravity, the coordinates of a test particle remain at rest. i.e. show that the geodesic equation,

$$\frac{d}{d\tau}U^\alpha + \Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu = 0 \tag{3}$$

is:

$$\frac{d^2x^i}{d\tau^2} = 0$$

Hint: use the metric

$$ds^2 = -dt^2 + \{1 + Asin\,[\omega(t - z)]\}dx^2 + \{1 - Asin\,[\omega(t - z)]\}dy^2 + dz^2 \tag{4}$$

Exercise 3.

If we have two particles, one of them at $x = 0$ and the other at $x = \epsilon$, both at $y = z = 0$, show (make all the calculations) that the proper distance is given by:

$$\begin{aligned}\Delta l \equiv \int |ds^2|^{1/2} &= \int |g_{\alpha\beta}dx^\alpha dx^\beta|^{1/2} \\ &\approx \left[1 + \frac{1}{2}h^{TT}_{xx}(x = 0)\right]\epsilon\end{aligned} \tag{5}$$

Calculate the value explicitly for a metric of the form (4).

Exercise 4.

Changing coordinates for metric (4) in exercise 2 to:

$$X = \{1 + (1/2)Asin(\omega t)\}x \tag{6}$$

$$Y = \{1 - (1/2)Asin(\omega t)\}y \tag{7}$$

Show that a ring of particles, located in the plane perpendicular to the wave propagation, is deformed harmonically into an elipse as the wave passes. Calculate the semimajor and semiminor axes in terms of the relevant metric coefficient.

Exercise 5.

Show that the cross polarization ($\times$) can be obtained through the following coordinate transformation

$$x \to \bar{x} = \frac{1}{\sqrt{2}}(x + y), \qquad y \to \bar{y} = \frac{1}{\sqrt{2}}(x - y) \tag{8}$$

apply to the metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + 2[1 - \epsilon h_{xy}(t - x)]dxdy + dz^2. \tag{9}$$

Exercise 6.

Assuming a metric:

$$ds^2 = -dt^2 + \{1 + Asin\,[\omega(t - z)]\}dx^2 + \{1 - Asin\,[\omega(t - z)]\}dy^2 + dz^2 \tag{10}$$

calculate the deviation $\delta x^i(\lambda)$ for a photon that communicates between two test particles.
Remember that for photons, $ds^2 = 0$.

Exercise 7.

Using null coordinates:

$$u = t - z \tag{11}$$

$$v = t + z \tag{12}$$

the following solution represents a plane wave :

$$ds^2 = -dudv + f^2(u)dx^2 + g^2(u)dy^2 \tag{13}$$

where f and g will need to be determined and we expect the solution to be a forward moving so we expect it to be only functions of u. Show that the only non-vanishing connection coefficients are (dot means derivative respect to u);

$$\Gamma^x{}_{xu} = \dot{f}/f, \qquad \Gamma^y{}_{yu} = \dot{g}/g$$

$$\Gamma^v{}_{xx} = 2\dot{f}/f. \qquad \Gamma^v{}_{yy} = 2\dot{g}/g$$

$$R^x{}_{uxu} = -\ddot{f}/f, \qquad R^y{}_{uyu} = -\ddot{g}/g,$$

and that Einstein's vacuum field equations vacuum reduce to:

$$\ddot{f}/f + \ddot{g}/g = 0 \tag{14}$$