Introduction to General Relativity 2025
Lesson 1: Special Relativity

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1 Newtonian framework

An implicit concept in Newtonian Mechanics is the idea of absolute time: that is the existence of a time that is the same for all observers independently of their system of reference. The demise of the idea of an absolute space is somehow easily achieved (at least for a physics major) when studying Galilean transformations: We are familiar with the occasional impossibility, for a short instant of time, to distinguish which car is moving in which direction, at a light stop. Are we moving forward or is the other car moving backwards?

Essentially this is the physical impossibility of detecting between inertial frames of reference. There is no absolute system of reference but at least there is a privileged set of them and there seems to be an absolute time common to all of them. We'll see that a new type of relativity will bring down the notion of absolute time too and consequently the concept of simultaneity in physics.

1.1 Galilean Transformations

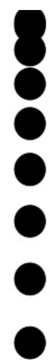
Newton's first law defines a privileged set of bodies, those not acted upon by forces; they move at constant velocity or they stay at rest. These set of bodies and the ones of co-moving observers are called inertial frames. Let's assume that one frame of reference O' has a constant velocity

$$\vec{v} = (v_x, v_y, v_z) \tag{1}$$

with respect to another called O. Then if the coordinates of an event at O' are related to the ones at O by the equations:

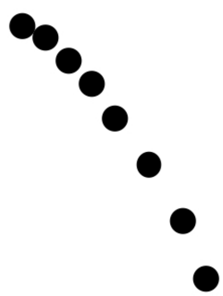
$$x = x' + v_x t; \quad y = y' + v_y t; \quad z = z' + v_z t; \quad t = t' \tag{2}$$

(Notice that the case $v_z = v_y = 0$ is what it's called **standard configuration** in some books.)
The last of equations in (2) means that there is an absolute time. Equation 2 give the coordinate translation needed to understand the different perception in the paradox of the cannonball thrown from the top of a sailboat's mast. In Galileo's book, the Dialogue of two New Sciences, the characters in the book recalled the Aristotelian impossibility of reconciling the two different views that two observers have of the following phenomenon: A cannon ball is thrown from a sailing boat that is moving a constant speed respect to the shoreline. The sailor throwing the ball sees:



A cannon ball as perceived by the sailor who threw it
from the top of the mast of a moving ship.

An observer on the beach sees the picture below as the ship movings in front of her.



The same cannon ball as perceived by an observer on the beach.

Both pictures are correct, as we know, and the descriptions can be easily transformed into one another using equations (2).

2 The principle of special relativity

The previous discussion formulates the impossibility of distinguishing between inertial frames of reference and makes the motion of one with respect to other a relative concept. Position and velocity are relative concepts. It took humankind 2000 years to shed the sacred concept of an absolute space: from the rigid hierarchical universe of Aristotle to the physical description of Galileo and Newton. Remember that in Aristotle's cosmogony there was an absolute privileged state: the one at rest. There was a natural position for each body according to its nature (things with more "earth" will quickly move to the ground, with more "air" will elevate). Motion occurs because of the need for each object to go back to its natural state after external agents changed it. After the Newtonian revolution it will take another half millennium to dispose of the concept of an absolute time. Newton's mechanics was the first successful unification in the history of physics: it unified the mechanics of the heavens with the mechanics of bodies on Earth. The second one was Maxwell's formulation of his equation of electromagnetism in 1865. This theory put electricity and magnetism as manifestations of the same phenomenon and predicted the existence of electromagnetic waves traveling at 299,792 kilometers per second. But Maxwell's theory was bringing about a contradiction with Galilean Relativity: Maxwell's equations implied that the speed of light ought to be the same (in vacuum) regardless of the relative velocity of the reference systems in motion utilized to measure it.

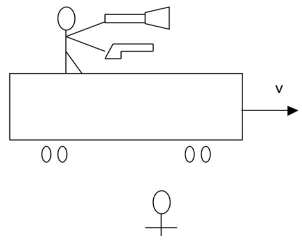
The so-called restricted principle is a natural consequence of the mathematical formulation of Galilean Relativity: Newton's laws are invariant under a Galilean transformation. The inclusion of all physical laws, encompassing Maxwell's equations as well, within the Principle of Special Relativity is Einstein's main reformulation of Relativity.

Postulate I

All inertial observers are equivalent. In other words; the physics described by all inertial observers is the same. In the modern language of relativity: the laws of physics are invariant under a Galilean transformation.

Postulate II

The velocity of light is the same in all inertial systems. This second statement is not obvious at all. Furthermore it is counterintuitive and in principle in contradiction with our understanding of Galilean Relativity. Think about the classical "gedanken experiment": We have the following two inertial frames: a train moving at constant speed v with respect to the ground. One of the observers travel on the top of one of the wagons and holds in his hands a pistol and a flashlight.



A flashlight and a gun are fired from a train in motion

A second observer is watching the train pass by from the ground. They are ready to perform all the needed experiments to determine positions, and velocities. The bodies under study are bullets from the pistol and the light from the flashlight. When the observer on the train shoots the gun the speed comes to be, with no surprise,$v + v_b$ as measured from the ground. When the experiment with the flashlight is performed the speed of light is the same for both observers!

2.1 The speed of light

Einstein himself claimed to have based his Special Relativity theory in two experiments: 1) Fizeau's experiment and 2) stellar aberration of light.

Fizeau's experiment

Fizeau measured the speed of light in a body in motion. To this effect he use an apparatus consisting in a tube with water circulating through it at a known velocity. He was expecting to measure a speed equal to the speed of light in vacuum plus the speed of water. The light beam was propagated in both directions (along the water and against it) to make the effect stronger by comparison between them. He used interferometric techniques similar to the ones used in LIGO. The unexpected effect for Fizeau was that the measured effect contradicted Galilean Relativity.

Stellar aberration

Stellar aberration is related to stellar parallax. Due to the motion of the Earth around the Sun the stars relative position in the sky changes throughout the year (see Figure). Parallax measurements is the only method to determine the distance to nearby stars (knowing the parallax angle and the baseline distance –Earth to the Sun– the distance to the star can be easily inferred by simple trigonometry). James Bradley —a British Astronomer Royale from the XVII century— set to measure the distance to the star Gamma Draconis. He observed that there was a substantial difference with the expected result: the angle did change, but the difference did not correspond to the Earth's orbital position. It was delayed:

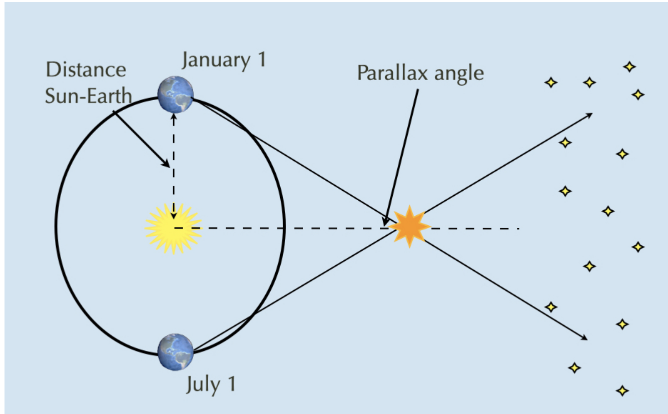


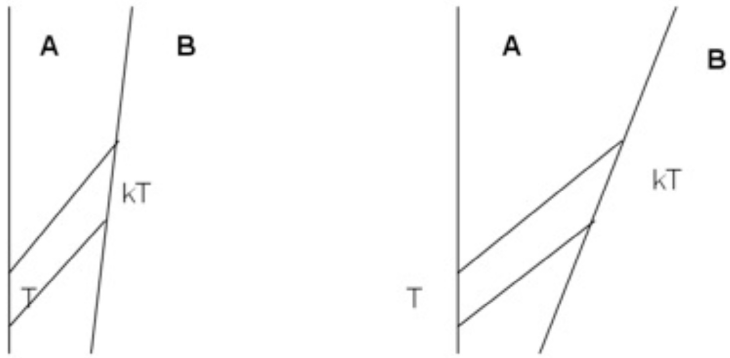
Figure 1: Distance to a nearby star.

this is the effect known as stellar aberration. What is the reason for this “aberration”. While light takes some time in reaching the Earth, our planet will keep moving while it arrives. Bradley managed to calculate the effect and verify that it was determined by the velocity of Earth in its orbit. The importance of this effect is that it is another proof that the speed of light is constant regardless of the relative speed of the stars respect to the Earth.

Another interesting fact related to the stars relative motion is that the observation of stars in a binary system shows that speed of light in vacuo is independent of the motion of the sources. If this was not true then we would nott perceive their orbits as Keplerian, i.e. circles and ellipses. Their orbits would appear distorted!

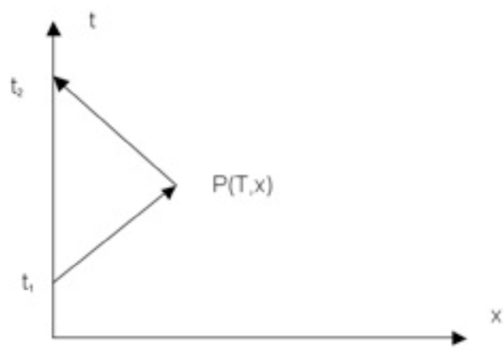
2.2 The k-factor

From our previous discussion it is clear then that intervals of time between events will change among inertial frames that are moving with respect to each other: if Postulate 2 is true then the times and lengths measured by inertial observers in relative motion will be different. We can assume that the difference can be proportional by a constant factor that only depends from the relative velocity. This is the k-factor. Notice that k obeys a reciprocal relationship for both observers. One observation: when we work with space-time diagrams a straight vertical line will represent a stationary observer. A tilted line (at somehow less than 45 degrees if t is the vertical axis) will represent an observer speeding away from the coordinate system. If you look at the figure:



Left: An observer B moving away at speed v from A.
Right: Observer B moving with speed $v' > v$ and the k factor is larger in this latter case.

To calculate the k factor it is useful to notice that the relationship between time and space traveled by a ray of light can be inferred from the following diagram. (notice that a light ray travels at $45°$ in a $t - x$ diagram as below (taking the system of units where $c = 1$.



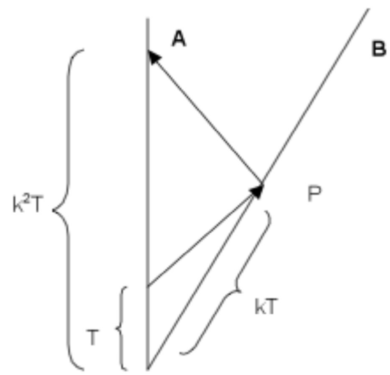
A light ray traveling from $x = 0$ and $t = t_1$ to point $P(T, x)$ and back to $x = 0$.

It is easy to convince yourself that the value of t and x for event P are given by:

$$(t, x) = \left(\frac{1}{2}(t_1 + t_2), \frac{1}{2}(t_2 - t_1)\right) \tag{3}$$

2.2.1 Relative speed of two inertial observers

Consider this scenario A sends a signal to B who is moving at a speed v away from A. A and B were "together" at t = 0. At time T later A send. T' is now equal to kT. The signal is bounced back from B to A. Then now for A T = kT' = k(kT). Looking at the picture:



A light ray traveling from $x = 0$ and $t = t_1$ to point $P(T, x)$ and back to $x = 0$.

The relationship is clear. Applying now this relationship to (3) where $t_1 = T$ and $t_2 = k^2 T$ we obtain:

$$(t, x) = \left(\frac{1}{2}(k^2+1)T, \frac{1}{2}(k^2-1)T\right) \tag{4}$$

Then

$$v = \frac{x}{t} = \frac{k^2-1}{k^2+1} \tag{5}$$

Solving for v we find

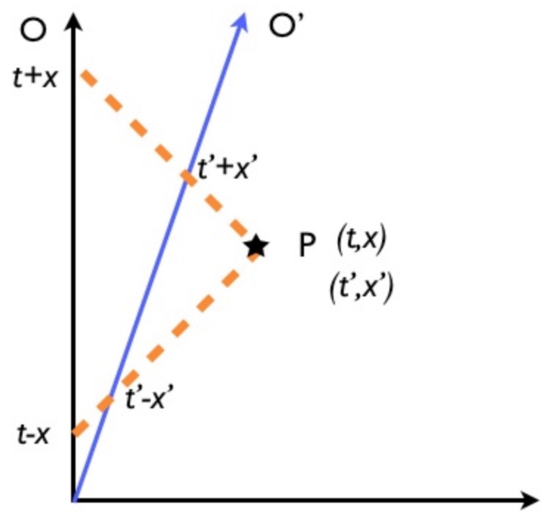
$$k = \left(\frac{1+v}{1-v}\right)^{\frac{1}{2}} \tag{6}$$

This a nice and compact formula that gives us the "dilation" factor in terms of the relative velocity of the observers. See that if $v = 0$ then $k \to 1$, which is what we expected; if $v \to -v$ then $k \to 1/k$. We shall go now directly into the derivation of the Lorentz transformations.

3 The Lorentz transformations

We already learn that Galilean transformations have an intrinsic flaw. They violate the constancy of the speed of light for all inertial observers. We will derive the transformations that relate coordinates and position of a given event for two inertial observers that are moving apart for each other.

We have an event P, which has coordinates (t, x) in A and coordinates (t', x') in B. We want now to relate t, x with t', x' (see figure below).



Observers O and O' and their coordinates.

To have a signal arriving at P at time t, A has to send the signal at a time $t - x/c$ (remember we use a unit system in which $c = 1$) and then receive it back at time $t + x/c$. Then from what we have learnt from k-calculus it is easy to see that:

$$t' - x' = k(t - x), \qquad t + x = k(t' + x') \tag{7}$$

Using the definition of k in terms of v, the speed of B relative to A, we can solve for t', x'. Just a little algebra shows: Adding the two equations:

$$2t' = t\left(\frac{1}{k} + k\right) + x\left(\frac{1}{k} - k\right) \tag{8}$$

Using (6) we have then:

$$t' = \frac{t - vx}{(1 - v^2)^{1/2}} \qquad \text{and} \qquad x' = \frac{x - vt}{(1 - v^2)^{1/2}} \tag{9}$$

This is called a *boost* in the x direction. It is simple to verify that:

$$t'^2 - x'^2 = t^2 - x^2 \tag{10}$$

We will learn later that this is an important invariant quantity. The fact that a Lorentz transformation has kept this quantity invariant is of tremendous importance. This quantity is called the interval and its mathematical significance is at the core of the geometric structure of space-time.

3.1 Newtonian Mechanics, Maxwell's equations and the need for a General theory of Relativity

Newtonian Mechanics is invariant under Galilean Relativity. If we change from a reference system $\{x\}$ to a reference system $\{x'\}$

$$x' = x - vt \tag{11}$$

Newton's equations

$$F = ma \tag{12}$$

remain the same because

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} = a \tag{13}$$

From this it follows that it is impossible distinguish rest from any uniform rectilinear motion. There is no preferred spatial position "x". (there is an absolute time though!). But Maxwell equations (and the Lorentz force) are not invariant under (11). Which is very clear from

$$\vec{F} = q(\vec{E} + \dot{\vec{r}} \times \vec{B}) \tag{14}$$

Which is v dependent. Actually the full relativistic Lorentz force is

$$\frac{d}{dt}\left(\frac{m\dot{\vec{r}}}{\sqrt{1-\left(\frac{\dot{\vec{r}}}{c}\right)^2}}\right) = q(\vec{E} + \dot{\vec{r}} \times \vec{B}) \tag{15}$$

Notice also that Coulomb's law

$$F = \frac{qq'}{r^2} \tag{16}$$

is not Lorentz invariant and it would violate Special Relativity. But...it is not a universal law. It is the low velocity limit of Maxwell's equations where the charges can be considered static or move slowly respect to one another. We can also see that Newton's law of Gravitation

$$F = \frac{mm'}{r^2} \tag{17}$$

would contradict Special Relativity if it were a Universal Law. It would require instantaneous "communication" between the masses if they were to move. But nothing can travel faster than light. Newton's law is a static, non-relativistic limit, valid only when the masses do not move rapidly respect to each other. We would need a field like the electromagnetic field to explain gravitation, of which Newton's gravity would be the low speed limit. We will see that we need a theory not limited to inertial systems of reference but one that also includes accelerated systems of reference.

4 The four dimensional world view

We can compare Galilean and Lorentz transformations from this table: Now we have a

Table 1: Galilean and Lorentz transformations compared

Galilean Transformation	**Lorentz Transformation**
$t' = t$	$t' = \frac{t-vx}{(1-v^2)^{1/2}}$
$x' = x - vt$	$x' = \frac{x-vt}{(1-v^2)^{1/2}}$
$y' = y$	$y' = y$
$z' = z$	$z' = z$

four dimensional continuum which we called space-time. In a Galilean transformation the quantity that is preserved, as can be easily seen[1] is:

$$\sigma = x^2 + y^2 + z^2 \tag{18}$$

which is the Euclidean distance. In a Lorentz transformation the preserved quantity is:

$$s^2 = t^2 - x^2 - y^2 - z^2 \tag{19}$$

This quantity is called a metric. A space-time for which this metric is invariant under Lorentz transformations is called a Minkowski space-time. We see then that the two postulates of special relativity imply that what we call Space has a completely different geometrical structure from the one we are used to deal with and to understand (the Euclidean).

[1] to visualize remember that the Euclidean geometry is defined by the distance between two points A with coordinates (x_1, y_1) and B with coordinates (x_2, y_2) being $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

5 Lorentz transformations revisited

We will use the two postulates of the special theory of relativity to deduce the Lorentz transformations. First if observer O sees a particle moving freely (i.e. no force acting on it) then O' should also see a free particle. This means that the trajectory of the observed particle should be a straight line in both systems of reference. Consequently because by the transformations -that transforms the particle's trajectory in one frame to another- straight lines remains straight lines, we required that our transformations be linear:

$$\vec{r} = \vec{r_0} + \vec{u}t \qquad \Leftrightarrow \qquad \vec{r'} = \vec{r'_0} + \vec{u'}t' \tag{20}$$

and linearity means:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = L \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{21}$$

with $y = y'$ and $z = z'$. Let's use now that the speed of light is the same in both inertial systems. Let's look at this quantity

$$I(t, x, y, z) = x^2 + y^2 + z^2 - c^2t^2 \tag{22}$$

Clearly I defines a sphere moving at the speed of light. If we look at a particular value of t like $t = t_0$ and we make $I = 0$ this would be a sphere of radius ct_0. In the primed system of reference we get:

$$I'(t', x', y', z') = x'^2 + y'^2 + z'^2 - c^2t'^2. \tag{23}$$

The spheres should be the same just because of the second postulate, If this is not clear think about this: The light travels at speed c in all inertial systems; consequently in S it travels a distance r (radially) after a time t ($c = r/t$). In the other system also the corresponding distance r' should be traveled in a time $t\prime$ such that $c = r'/t'$. then:

$$I = 0 \qquad \Leftrightarrow \qquad I' = 0 \tag{24}$$

which means:

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2. \tag{25}$$

With $y = y'$ and $z = z'$ we get then,

$$x'^2 - c^2t'^2 = x^2 - c^2t^2 \tag{26}$$

If we take L

$$\begin{pmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{pmatrix} \tag{27}$$

we get:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{28}$$

then

$$x' = \cosh(\psi)x + \sinh(\psi)ct \tag{29}$$

$$ct' = \sinh(\psi)x + \cosh(\psi)ct \tag{30}$$

If we make $x' = 0$ we know that $x = vt$. So

$$\cosh(\psi)x + \sinh(\psi)ct = 0 \tag{31}$$

From this we see that:

$$\tanh(\psi) = \frac{x}{ct} = \frac{v}{c} \tag{32}$$

And using that

$$\cosh^2(\psi) - \sinh^2(\psi) = 1 \tag{33}$$

solve for $\cosh(\psi)$:

$$\cosh^2(\psi) = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \tag{34}$$

Notice that this defines ψ. It is easy to use

$$\beta = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \tag{35}$$

And now it's easy to get (do it!):

$$\sinh^2(\psi) = \frac{-v}{c}\cosh(\psi) = -\frac{v}{c}\beta \tag{36}$$

The matrix L is then:

$$\begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{-v/c}{\sqrt{1-\frac{v^2}{c^2}}} \\ \frac{-v/c}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} \tag{37}$$

And the Lorentz transformation are:

$$t' = \beta\left(t - x\frac{v}{c^2}\right) \tag{38}$$

$$x' = \beta(x - vt) \tag{39}$$

$$y' = y \tag{40}$$

$$z' = z \tag{41}$$

5.1 Exercise

How does the Lagrangian of a free particle:

$$L = -\sqrt{1 - \left(\frac{dx}{dt}\right)^2} \tag{42}$$

transforms under the following coordinate transformation (q, τ):

$$q = \cosh(\psi)x + \sinh(\psi)ct \tag{43}$$

$$\tau = \sinh(\psi)x + \cosh(\psi)ct \tag{44}$$

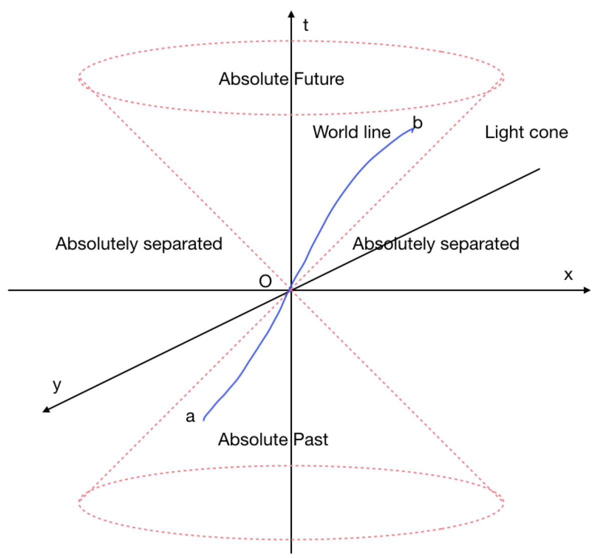
Discuss.

6 The principle of least action

This principle states that for all mechanical systems there exists a certain integral S, called the action, which has a minimum (or maximum) value for the "real life" path it follows in its motion, so that its variation δS is zero. To determine the action for a free particle (a particle not under the influence of any external force), this integral must not depend on our choice of reference system, that is, it must be invariant under Lorentz transformations. We assume that it must depend on a scalar. Furthermore, it is clear that the integrand must be a differential of the first order proportional to the distance that particle follows in space.This is called the *interval*. So we take αds, where α is some constant. So for a free particle the action must have the form:

$$S = -\alpha \int_a^b ds \tag{45}$$

where the integral is along the world line of the particle from point a at time t_1 to point b at time t_2.



The light cone and the world line
of a particle in a Minkowski diagram.

The constant α is characterizing the particle (an intrinsic property). The minus sign guarantees a minimum (and not a maximum). We will use now a new definition:

6.1 Proper time

Suppose that in a certain inertial reference system we observe clocks which are moving relative to us in an arbitrary manner. At each different moment of time this motion can be considered as uniform. Thus at each moment of time we can introduce a coordinate system rigidly linked to the moving clocks, which with the clocks constitutes an inertial reference system. In the course of an infinitesimal time interval dt (as read by a clock in our rest frame) the moving clocks go a distance $\sqrt{dx^2+dy^2+dz^2}$ what time interval dt' is indicated for this period by the moving clocks? In a system of coordinates linked to the moving clocks, the latter are at rest, i.e., $dx' = dy' = dz' = 0$. Because of the invariance of intervals

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^{2'}, \tag{46}$$

from where we can quickly see:

$$dt' = dt\sqrt{1 - \frac{dx^2+dy^2+dz^2}{c^2dt^2}} = dt\sqrt{1-\frac{v^2}{c^2}} = \frac{ds}{c}, \tag{47}$$

We can refer to the integral as

$$S = -\alpha\int_a^b ds = \int_a^b Ldt \tag{48}$$

where L represents the Lagrange function of our system. Using (28) we get

$$S = -\int_a^b \alpha c\sqrt{1-\frac{v^2}{c^2}}dt \tag{49}$$

and the Lagrangian for the free particle is:

$$L = \alpha c\sqrt{1-\frac{v^2}{c^2}} \tag{50}$$

What is α? In non-relativistic mechanics we know it is the mass of the particle. Let us find the relation between α and m. It can be determined from the fact that in the limit as $c\to\infty$, our expression for L must go over into the classical expression $L = mv^2/2$. We expand L in powers of v/c Neglecting the terms higher than $O(v^2/c^2)$

$$L = \alpha c\sqrt{1-\frac{v^2}{c^2}} \approx -\alpha/c + \frac{\alpha v^2}{2c} \tag{51}$$

The Lagrangian is defined up to a constant of motion which means

$$\alpha = mc \tag{52}$$

The action for a free particle then is

$$S = -mc \int_a^b ds \tag{53}$$

and the Lagrangian is:

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \tag{54}$$

7 Integrals of motion

We will see what are the integrals of motion in Relativistic Mechanics.

7.1 Noether's theorem

Let's assume that there is the following transformation from coordinates q_i, t to ${q_i}', t'$.

$${q_i}' = q_i + \epsilon \Psi_i(q,t) \tag{55}$$

$$t' = t + \epsilon X(q,t) \qquad \epsilon \to 0. \tag{56}$$

And that the following quantity remains invariant under this transformation:

$$\int_{t_1}^{t_2} L(q, \frac{dq}{dt}, t) = \int_{t'_1}^{{t'}_2} L(q', \frac{dq'}{dt'}, t') \tag{57}$$

Then

$$\sum_i \frac{\partial L}{\partial q_i} \left(\dot{q}_i X - \Psi_i \right) - LX \tag{58}$$

is an integral of motion (i.e. a constant). It is not difficult to see that the momentum of a particle is the conserved quantity associated with an invariance under coordinate displacements and it is given by the vector $\vec{p} = \partial L/\partial \vec{v}$ (where $\partial L/\partial \vec{v}$ is the symbolic representation of the vector whose components are the derivatives of L with respect to the corresponding components of $\vec{v}$). Using (35) we get:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{59}$$

For small velocities compared to $v \ll c$ we recover the classical definition. The acceleration is

$$\frac{d\vec{p}}{dt} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt} \tag{60}$$

The energy of the particle is the conserved quantity associated with an invariance under time displacement:

$$\mathcal{E} = \vec{p}\cdot\vec{v} - L \tag{61}$$

Using (35) and (40)

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{62}$$

This crucial formula shows, that in relativistic mechanics the energy of a free particle does not go to zero for $v = 0$, but

$$\mathcal{E} = mc^2 \tag{63}$$

This quantity is called the rest energy of the particle. For small velocities ($v/c \ll 1$), we have, expanding (62) in series in powers of v/c,

$$\mathcal{E} \approx mc^2 + \frac{1}{2}mv^2 \tag{64}$$

Formula (44) is valid for any body which is at rest as a whole. We call attention to the fact that in relativistic mechanics the energy of a free body (i.e. the energy of any closed system) is a completely definite quantity which is always positive and is directly related to the mass of the body. In this connection we recall that in classical mechanics the energy of a body is defined only to within an arbitrary constant, and can be either positive or negative. The energy of a body at rest contains, in addition to the rest energies of its constituent particles, the kinetic energy of the particles and the energy of their interactions with one another. In other words, mc^2 is not equal to $\sum_a m_a c^2$ (where m_a are the masses of the particles), and so m is not equal to m_a. Thus in relativistic mechanics the law of conservation of mass does not hold: the mass of a composite body is not equal to the sum of the masses of its parts. Instead only the law of conservation of energy, in which the rest energies of the particles are included, is valid. Squaring (40) and (43) and comparing the results, we get the following relation between the energy and momentum of a particle:

$$\frac{\mathcal{E}^2}{c^2} = p^2 + m^2c^2 \tag{65}$$

Which is the Hamiltonian ($\mathcal{H}$) of the system.

$$\mathcal{H} = c\sqrt{p^2 + m^2c^2} \tag{66}$$

For low velocities, $p \ll mc$, and we have approximately

$$\mathcal{H} \approx mc^2 + \frac{1}{2m}p^2 \tag{67}$$

i.e., except for the rest energy we get the familiar classical expression for the Hamiltonian. From (40) and (43) we get the following relation between the energy, momentum, and velocity of a free particle:

$$\vec{p} = \mathcal{E}\frac{\vec{v}}{c^2} \tag{68}$$

For $v = c$, the momentum and energy of the particle become infinite. This means that a particle with mass m different from zero cannot move with the velocity of light. Nonetheless, in relativistic mechanics, particles of zero mass moving with the velocity of light can exist. From (49) we have for such particles:

$$p = \frac{\mathcal{E}}{c} \tag{69}$$

The same formula also holds approximately for particles with nonzero mass in the so-called ultrarelativistic case, when the particle energy $\mathcal{E}$ is large compared to its rest energy mc^2. Let's extend our formalism to 4 dimensions.

$$\delta S = -mc\delta \int_a^b ds = 0 \tag{70}$$

To find an expression for dS we noticed that

$$ds^2 = -cdt^2 + dx^2 + dy^2 + dz^2 \tag{71}$$

We will use the following convention: $(x_0, x_1, x_2, x_3) = (-ct, x, y, z)$. This means that ds can be written the following way:

$$ds = \sqrt{-cdt^2 + dx^2 + dy^2 + dz^2} = \sqrt{dx_\alpha}\sqrt{dx^\alpha} \tag{72}$$

and $dx_\alpha = (x_0, x_1, x_2, x_3)$ while $dx^\alpha = (x^0, x^1, x^2, x^3)$. Repeated index once as sub the other as upper imply summation (Einstein convention). We will learn later what is the essential difference between indices up and down. With this:

$$\delta S = -mc \int_a^b \frac{dx_\alpha \delta dx^\alpha}{ds} = -mc \int_a^b u_\alpha \delta dx^\alpha \tag{73}$$

where u_α is the velocity in the direction α. Integrating by parts we get:

$$\delta S = -mcu_\alpha \delta x^\alpha |_a^b + mc \int_a^b \delta x^\alpha \frac{du_\alpha}{ds} ds \tag{74}$$

to get the equations of motion we compare different trajectories between fixed two points, which means $(\delta x^\alpha)_a = (\delta x^\alpha)_b = 0$. The actual trajectory is then determined from the condition $\delta S = 0$. From (55) we get $du_\alpha/ds = 0$. This is a constant velocity for the free particle in four-dimensional form. If we let the action vary as a function of the coordinates $(\delta x^\alpha)_a = 0$ and we can let the final point vary but constrained to satisfying the equation of motion. This means that simplifying $(\delta x^\alpha)_b$ as just δx^α.

$$\delta S = -mcu_\alpha \delta x^\alpha \tag{75}$$

From this we obtain the four vector

$$p_\alpha = -\frac{\partial S}{\partial x_\alpha} \tag{76}$$

which is the momentum four-vector. The partial derivative respect to the space coordinates are the traditional moment in 3-dimensions. The time derivative is the energy of the particle which is the time component of the momentum. It is customary then to write the 4 momentum. the covariant components (the vector with the indices down) are

$$p_\alpha = (\mathcal{E}, -p_i) \tag{77}$$

and the contravariant components are

$$p^\alpha = (\frac{\mathcal{E}}{c}, p^i) \tag{78}$$

where we use the greek index α when we refer to the 4 dimensions (3 space, 1 time). The latin i we reserve it just for the space components (like x, y, z). from (56) the components of the 4-momentum are

$$p^\alpha = mcu^\alpha \tag{79}$$

If we take the four velocity as $u^\alpha = dx^\alpha/ds$

$$u^\alpha = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{u^i}{c\sqrt{1 - \frac{v^2}{c^2}}} \right) \tag{80}$$

where the first term is the time component of the velocity and the second one are the three space components of the standard 3-velocity. Substituting (61) in (60) we get:

$$p^\alpha = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{mu^i}{\sqrt{1-\frac{v^2}{c^2}}} \right) \tag{81}$$

Thus, in relativistic mechanics, momentum and energy are the components of a single four-vector. From this we could get the formulas for transformation of momentum and energy from one inertial system to another.

8 Mathematical Properties of Lorentz transformations

1. We can think in terms of imaginary time coordinate: $T = it/c$, this would be one way of interpreting the $-$ sign in front of the time coordinate. And remembering this relationship: $\sinh(x) = i\sin(ix)$ We can then interpret the hyperbolic transformations as euclidean rotations in complex space.

2. If v is very small we recover Galilean transformations.

3. Solving for the unprimed coordinates what we would see is similar formulas as if we take -v for v and reverse the priming.

4. Special Lorentz transformations form a group. A group is an algebraic structure consisting of a set together with an operation that combines any two of its elements to form a third element that is also a member of the set. In addition, the set and the operation must satisfy the following properties: associativity, identity and invertibility. The Lorentz transformations is associative, i.e. Two successive Lorentz transformations yield another Lorentz transformation, it has an identity element, the Lorentz transformation with 0 velocity, and it has an inverse, the Lorentz transformation with velocity -v does give back the original system without transformation.

5. The line element: $ds^2 = c^2 - dx^2 - dy^2 - dz^2$ is invariant under a Lorentz transformation. It is the square of the interval between events that are infinitesimally close (The Minkowkski metric).

9 Transformation of velocities

(*This section follows the approach taken in Ray D'Inverno's book)*

The main result here is that obviously the velocities "seen" by different inertial observers, who are in motion with respect to each other, differ. i.e. taking differentials in (19)-(22)we get:

$$dt' = \beta \left(dt - dx \frac{v}{c^2} \right) \tag{82}$$
$$dx' = \beta(dx - vdt) \tag{83}$$
$$dy' = dy \tag{84}$$
$$dz' = dz \tag{85}$$

then,

$$u'_1 = \frac{dx'}{dt'} = \frac{u_1 - v}{1 - u_1 v/c^2} \tag{86}$$

$$u'_2 = \frac{dy'}{dt'} = \frac{u_2 - v}{\beta(1 - u_2 v/c^2)} \tag{87}$$

$$u'_3 = \frac{dz'}{dt'} = \frac{u_3 - v}{\beta(1 - u_3 v/c^2)} \tag{88}$$

10 Acceleration in special relativity

Let's calculate the inverse transformation of (86),

$$u_1 = \frac{u'_1 + v}{1 + u'_1 v/c^2} \tag{89}$$

And from it we calculate the differential

$$du_1 = \frac{du'_1 + v}{1 + u'_1 v/c^2} - \left(\frac{u'_1 + v}{(1 + u'_1 v/c^2)^2} \right) \frac{v}{c^2} du'_1 = \frac{1}{\beta^2} \frac{du'_1}{(1 + u'_1 v/c^2)^2} \tag{90}$$

And in the Lorentz transformation we similarly calculate the inverse

$$t = \beta(t' + x'v/c^2), \tag{91}$$

and obtain

$$dt = \beta(dt' + dx'v/c^2) = \beta(1 + u_1'v/c^2)dt'. \quad (92)$$

Using then (91) and (92) we can calculate how the acceleration in the x_1 direction transforms:

$$\frac{du_1}{dt} = \frac{1}{\beta^3(1 + u_1'v/c^2)^3}\frac{du_1'}{dt'}. \quad (93)$$

Similarly

$$\frac{du_2}{dt} = \frac{1}{\beta^2(1 + u_1'v/c^2)^2}\frac{du_2'}{dt'} - \frac{vu_2'}{c^2\beta^2(1 + u_1'v/c^2)^3}\frac{du_1'}{dt'} \quad (94)$$

and

$$\frac{du_3}{dt} = \frac{1}{\beta^2(1 + u_1'v/c^2)^2}\frac{du_3'}{dt'} - \frac{vu_3'}{c^2\beta^2(1 + u_1'v/c^2)^3}\frac{du_1'}{dt'} \quad (95)$$

Notice that this just means: if a particle is accelerated in one inertial system it is accelerated for all inertial systems. Of course the value is not invariant. It changes from one system to another depending on the relative velocities of them respect to each other. If its acceleration is zero in one system then it's zero in all other inertial systems.

The following is a proof of the fact that the acceleration is an absolute quantity. We can write the equations for the time derivatives of the velocities expressed in (93)-(95) the following way:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ B & C & 0 \\ D & 0 & C \end{pmatrix} \begin{pmatrix} {a_1}' \\ {a_2}' \\ {a_3}' \end{pmatrix} \quad (96)$$

where

$$a_i = \frac{du_i}{dt} \qquad \text{and} \qquad {a'}_i = \frac{d{u'}_i}{dt'} \quad (97)$$

A and C are functions of u_1' , B and D are functions of u_2' and u_3' as well respectively. The determinant of the transformation is AC^2. Notice by simple inspection of the equations that it's never 0, even if some of the accelerations are zero. So the transformation has an inverse. Consequently if the accelerations are zero in one system they remain zero for all other systems. If they are different than zero, then they will remain different from zero, although the values will be different.

To summarize let's compare the theories:

Theory	Position	Velocity	Time	Aceleration
Newtonian	Relative	Relative	Absolute	Absolute
Special Relativity	Relative	Relative	Relative	Absolute
General Relativity	Relative	Relative	Relative	Relative

10.1 Uniform acceleration

From Newton's first law we know that a body moving under uniform acceleration has

$$\frac{du}{dt} = constant \tag{98}$$

This could be misleading. Given enough time it looks that the magnitude of the velocity can increase linearly with time without limit which would contradict the physics underlying special relativity. A way around this is **to adopt a different definition for uniform acceleration**. Due to the fact that the actual value is relative we can say that the acceleration of a particle is uniform if an observer in an instantaneously co-moving frame measures that same value. If the velocity of the accelerated particle in question is $u = u(t)$ relative to an inertial system of reference S, then at any instant in the **comoving** system S', v, the velocity of S' respect to S, is $v = u$, or equivalently u', the velocity of the particle in S' is $u' = 0$.
Starting with (86) we can calculate u and u' :

$$u = \frac{u_1' + v}{1 + u'v/c^2} \tag{99}$$

the differential is

$$\begin{aligned} du =& \frac{du_1' + v}{1 + u'v/c^2} - \left(\frac{u' + v}{(1 + u'v/c^2)^2} \right) \frac{v}{c^2} du' \\ =& \frac{1}{\beta^2} \frac{du'}{(1 + u'v/c^2)^2} \end{aligned} \tag{100}$$

The time differential can be found from the Lorentz transformation for it:

$$t = \beta(t' + x'v/c^2), \tag{101}$$

$$dt = \beta(dt' + dx'v/c^2) = \beta(1 + u'v/c^2)dt' \tag{102}$$

From where we get the component of the acceleration in the direction of motion as measured by the two systems are related in this manner

$$\frac{du}{dt} = \frac{1}{\beta^3(1 + u'v/c^2)^3}\frac{du'}{dt'} \tag{103}$$

We see that from here we can get that the acceleration in the frame S is then:

$$\frac{du}{dt} = \frac{1}{\beta^3}a = \left(1 - \frac{u^2}{c^2}\right)^{3/2} a \tag{104}$$

Integrating the differential equation using separation of variables:

$$\frac{du}{\left(1 - u^2/c^2\right)^{3/2}} = adt \tag{105}$$

Assuming that the particle starts from rest at $t = t_0$

$$\frac{u}{\left(1 - u^2/c^2\right)^{1/2}} = a(t - t_0) \tag{106}$$

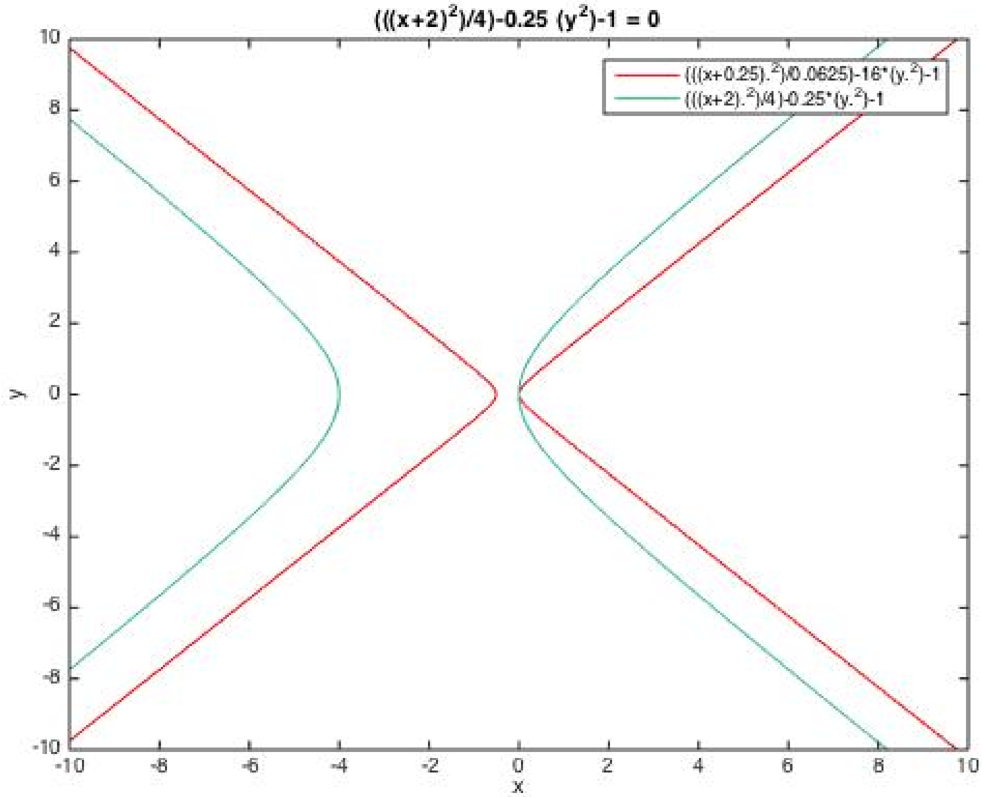
and solving now for $u = dx/dt$

$$\frac{dx}{dt} = \frac{a(t - t_0)}{[1 + a^2(t - t_0)^2/c^2]} \tag{107}$$

which integrated respect to t gives:

$$(x - x_0) = \frac{c}{a}\left(c^2 + a^2(t - t_0)^2\right)^{1/2} - \frac{c^2}{a} \tag{108}$$

The following is a plot done in Mathematica: Notice that we graph two curves, one with acceleration $a = 4$ and the other one with acceleration $a = 0.5$ (the dashed line). c is taken to be 1, and x_0 and t_0 are taken both zero.



Event horizons for two observers,
one with $a = 4$, green curve and the other with $a = 0.5$, red curve.

The lines define event horizons. Event horizons are surfaces for which the escape velocity is equal to the speed of light. In other words they are trapping surfaces from where nothing can escape. Notice that they appear here because of the accelerated nature of the particle. These event horizons provoked some heated debate in cosmology. Event horizons can radiate through what is known as the Hawking effect. The radiation would show as a swarm of particles created in vacuum that observers would see just as a result of being accelerated. At the same time the horizons would disconnect them from communicating with some regions of the universe. This is posing some "ignorance" problem. Hawking speculated in the 70's that this effect would make very difficult an absolute definition of what elementary particles are.

11 Some reflections about the Special Theory and the Road to the General one

(excerpts from "The Sounds of the Cosmos" by M. Díaz, J. Pullin, G. Gonzalez to be published by MIT Press)

The two postulates of the special theory, in spite of their simplicity, completely changed fundamental concepts of the physical world and affected other areas of knowledge, including philosophy.

Time, —the quintessential measure of change—, now was observer-dependent and was no more an absolute concept. Absolutes have had great importance for human beings, since they provide certainties. The questioning of absolutes has a shaking effect on human belief systems. The impact of Einstein's theory was enormous and attracted the interest of intellectuals in very different fields of knowledge. On April 6th 1922 the French Society of Philosophy invited Einstein to speak about relativity. An important debate took place with the French philosopher Henri Bergson. At the time Bergson was more famous than Einstein and there is speculation that Einstein did not get the Nobel Prize for the theory of relativity due to Bergson's opposition (Einstein received the Nobel prize in 1921 for his work on the photoelectric effect). Many consider that Bergson did not understand Einstein's theory. In a certain sense this implied a parting of the waters between physicists and philosophers in the subsequent years.

There is no doubt that indeed the new theory of special relativity introduced a new way of thinking —at least in physics—. It allowed to apply the methods of classical mechanics to electromagnetism. And with this it produced a new mechanics: the relativistic one. A result from it —that Einstein himself anticipated— is the equivalence between mass and energy, which states that the energy of a body at rest is equal to its mass multiplied by the speed of light squared (the super famous $E = mc^2$). Since the speed of light is enormous, even a stationary body with a small mass has a considerable amount of energy: this is known as its rest energy. Einstein proposed this equivalence between mass and energy in an article published in 1905 titled "Does the inertia of a body depend on its energy content?".

Near the end of the article Einstein mentions that the prediction could be tested with a body that loses energy by radioactive emission[2]. Radioactive disintegration is the process by which an unstable atomic nucleus loses energy emitting electromagnetic radiation. Measuring the emitted energy by a radioactive compound, like uranium salts, and

[2]Radioactivity was discovered by Antoine Henri Becquerel in 1896. He received the Nobel Prize for this discovery in 1903.

weighting its mass before and after the emission, the formula $E = mc^2$ could be verified experimentally.

According to Pais (the Einstein biographer we introduced in the previous chapter), Einstein wrote in 1905 to his friend, the Swiss mathematician Conrad Habicht, about this article: “This line of thought is... fascinating, but I cannot know if the dear Lord, playing a prank on me, laughs about me with this discovery”. At that moment Einstein could not imagine the incredible consequences that would be derived.

More than 30 years later —in 1939—, Einstein sent President Franklin Delano Roosevelt a famous letter in which he mentioned that the technology to start a nuclear chain reaction was in the hands of the Nazis in Germany. He argued with him that the US government should invest money and support the development of an atomic bomb in the US. On July 16th 1945 the first test of this bomb took place in the desert in a site known as Jornada del Muerto, 20 miles southeast of Socorro, New Mexico. The energy liberated in the explosion —equivalent to 21,000 tons of TNT—, confirmed soundly the principle of equivalence of mass and energy predicted by Einstein, and started a new era in the history of mankind: the atomic one. Einstein would later regret sending that letter.

The period that goes from the development of special relativity to the formulation of general relativity (1905 to 1914) was turbulent. These were trying times for humankind: on July 28 1914 the first world war started and would become one of the longest and deadliest in history. Its implications went well beyond physics.

Pais calls the task of understanding what was going through the mind of the thinkers that were changing the way of viewing the universe “the edge of history”. Pais worked with Einstein at the Institute for Advanced Study in Princeton, and had the chance to explore closely the sharpness of this edge. He tells that when he himself asked Einstein about the evolution of thinking by scientists like Lorentz, Poincare and others during this transition time, he replied that the birth of special relativity was “den Schritt”: the step, in German. For Einstein, the development of special relativity —a revolutionary event in the history of human thought— with multiple scientific and cultural impacts, was only that: a first step in the development of a more complete theory.

Einstein envisioned his development of the theory of relativity within a unified theory that could explain the universe in its entirety. A theory that integrated all the fundamental ideas of physics in a single and elegant formula: a “theory of everything”, as it was called by Richard Feynman, the great American physicist who won the Nobel prize in 1965.

Einstein made several attempts from 1905 to 1907 to integrate gravity with the theory of relativity. The theory of general relativity as we know it today was presented in 1915. These ten years from 1905 to 1915 were for Einstein a period of transition, that reflected also in his professional life. He passed the exam that allowed him to teach, and he accepted a job as a college professor, leaving his staff position at the patent office.

12 The general theory of relativity

12.1 The next steps

In Newton's theory, gravity propagates at infinite speed. But the theory of special relativity says that nothing can be faster than light. There was therefore a contradiction between both theories. Einstein continued studying how Newton's theory should be modified — in particular the gravitational force— to incorporate the results of special relativity. The problem was a hard one and Einstein attacked it in phases.

The "happiest thought" of his life

In a talk that Einstein gave in Kyoto on December 14th 1922, he described an episode that he recalled as taking place in 1907: "I was sitting in a chair in my patent office in Bern and the following occurred to me: if a man falls freely he would not feel his own weight. I was surprised that this simple imaginary experiment had in me such a big impact". He would later refer to this thought as the happiest in his life.

This example shows in which way an experiment —even an imaginary one—, can lead to a theoretical conclusion. Galileo was perhaps the first modern physicist to imagine experiments. These *thought experiments* consist in extracting the fundamental aspects of a real and complex physical situation and "performing" the experiment with the imagination in a simplified situation. For instance, Galileo imagined a physical space without air friction to understand the fundamental aspects of the free fall of bodies. Einstein used thought experiments like Galileo: to figure out the theoretical principle it was not necessary to jump with a scale glued to one's feet from an airplane. Being able to think about the experiment and to extract the correct scientific conclusions from it, is an act of intellectual bravery.

Precisely these thought experiments were crucial for the development of the general theory of relativity. In 1907 Einstein submitted a review article about the theory for publication in the journal *Yearbook of Radioactivity and Electronics*. In this article he laid out three themes that he considered important and that were decisive to move forward in the road towards the general theory: the principle of equivalence, the gravitational redshift and the deflection of light.

The equivalence principle

The principle of equivalence lies at the foundations of the general theory of relativity. It basically states —along the lines of Einstein's "happiest thought"—, that a system with constant acceleration is indistinguishable from one in a uniform gravitational field. We have just introduced for the first time the idea of a "field", a very important concept in physics. The reader may have heard about "electromagnetic fields", using the same idea. A field is a quantity that is defined at every point in space and time (space-time!). For example, wind can be described as a velocity of air at every point, and then it is a "field". For an electric field, we define it as the electric force that would act on an electric charge at every point, even if there is no actual electric charge (the field is produced by other electric charges). Of course *if* we put an electric charge at that point, the force on it will be determined by the field we calculated. Similarly, the gravitational field produced by a star or a planet (or any mass) is the gravitational force it would produce on a mass at some distance from it, even if there is no mass there.

A uniform field is one that has the same magnitude and direction everywhere. When we are on the surface of the Earth (a state in which we spend pretty much all our lives) we feel the gravitational field of the Earth. It varies little around us because it depends on the distance to the center of the Earth and that does not change too much even if we move to the top of a mountain, so we can consider it for all practical purposes as a "uniform gravitational field".

Einstein's happiest thought can be understood with a simple experiment. Let's imagine a rocket that travels to outer space, far away from the Earth or any planet or stellar object, where the gravitational field has a negligible magnitude. Moreover, the rocket is moving with an acceleration equal to the one an object falling on the surface of Earth would have (the acceleration Galileo measured). If a person weighed herself in a scale on the floor of the rocket, she would measure the same "weight" as on Earth: that is, could not distinguish if she is in an accelerated rocket in outer space or in a rocket parked on Earth if there are no windows in the rocket (see the illustration in figure 2).

The gravitational redshift

"Redshift" is the name given to the lowering of frequencies of waves (this is because the waves that constitute visible light become redder if their frequency becomes lower).

We experience shifts in frequencies due to relative motion of objects: for instance, the change in the sound frequency of an ambulance or a firetruck siren as it moves away or towards us can be clearly perceived by our ears. When the ambulance is approaching us, the sound waves emitted by the car are reaching us more often than when emitted, so the frequency (pitch) is higher. When the ambulance recedes, the pitch is lower because the waves reach us less frequently.

A less familiar example is police radar. Police checks the speed of cars using a transmitter of electromagnetic radiation (usually microwaves or radio waves) combined with a receiver. When this device is pointed to a car, it sends an electromagnetic signal that bounces off it and is captured backed by the receiver. The device compares the frequency of both the emitted wave as well as the reflected one. If the car is not moving, both fre-

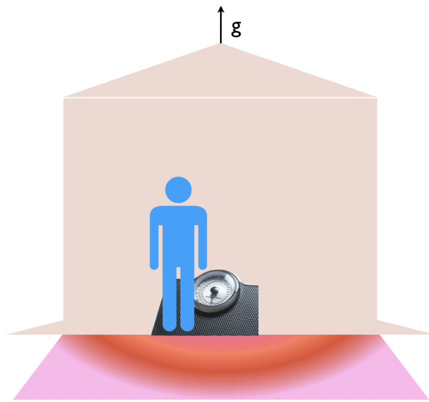


Figure 2: Acceleration at the Earth and in a rocket. A person weighing herself on a scale inside a rocket in outer space moving with acceleration equal to Earth's gravity (g), weighs the same as on Earth.

quencies are the same. But if the the car is moving, the reflected wave frequency changes, and the change is a measure of the car's velocity: this is the operating principle of the Doppler radar, and the reason for many speeding tickets.

What does redshift have to do with gravity? Einstein wanted to extend special relativity to accelerated systems. He was in particular interested in accelerated systems in which the speed changes at a constant rate like with the acceleration of gravity. Let us think about two observers at some distance from one another who are accelerated in the same direction: we can even think that both have the same (constant) acceleration which is equal to the acceleration of gravity. If the observer who is behind sends an electromagnetic signal to the other observer (turning on a flashlight, for example), she will receive the signal after some time but then her velocity will have changed (because she is accelerated), so there is a relative velocity between the emitter (the flashlight) and the observer. This means that the signal received will be shifted to lower frequencies: it will be redshifted.

Einstein concluded that observers who are accelerated would measure a shift in the frequency of the light. But according to the principle of equivalence accelerated systems of reference are indistinguishable from a gravitational field. From this Einstein concluded that the light will shift its frequency in the presence of a gravitational field like the Earth's.

The light changes frequency, shifting more to the red, the higher the acceleration or the more intense the gravitational field. Robert Pound and his graduate student Glen A. Rebka Jr. performed an experiment verifying the gravitational redshift prediction in 1959 (again, decades after the prediction!). The experiment consisted in using a gamma ray source emitting from the top of a 74 ft tower and measuring the change of its frequency with a receiver positioned at its bottom.

Light deflection

Einstein also predicted another consequence of the principle of equivalence for effects of gravity on light rays, in addition to the redshift. In the presence of a gravitational field light does not travel in a straight line. Its path bends.

This had been already considered by other scientists earlier, assuming that gravitational forces acted on light beams like it did on massive objects. Johann Georg von Soldner, a German physicist, mathematician, astronomer, published in 1801 an article titled “On the Deflection of a Light Ray from its Rectilinear Motion”, calculating what we call now Newtonian bending of light. The light deflection produced by the Sun would be about 0.8 arc seconds as seen from Earth (an arc second is 1/3600 of one degree). This is for a ray that just misses the sun (the effect is larger the closer to the Sun the ray passes. In 1911, Einstein revisited the effect, and expressed strong interest in an experimental proof of this effect: “It is greatly to be desired that astronomers take up the question broached here, even if the considerations here presented may appear insufficiently substantiated or even adventurous. Because apart from any theory, we must ask ourselves whether an influence of gravitational fields on the propagation of light can be detected with currently available instruments.” Later on we will discuss such experimental attempts. Remarkably, the correct calculation in the general theory of relativity of the effect (not available in 1911) predicts twice the magnitude for it, due to the curvature of space.

12.2 Physics calls geometry for help

After the early approaches to a relativistic theory of gravity, Einstein did not work much on the subject till 1911, when he took a job as professor at the University of Prague. Einstein’s papers published in 1907 assumed space was still considered flat: the familiar space that scientists call “Euclidean”, where parallel lines never cross. By 1912 Einstein convinced himself that he needed to search for a new gravitational dynamics and understood that the laws of Euclidean geometry that the Greeks had developed and had been used for 2000 years were no good for this purpose.

Paul Ehrenfest —an Austro-Hungarian physicist who carried out great part of his career at the University of Leiden, in the Netherlands, and met Einstein in 1910 in Prague—, had formulated a paradox that greatly influenced Einstein. In a thought experiment, he applied the principles of the special theory of relativity to a rotating system of reference. The reader probably remembers our couple María and José experiencing the Galilean addition of velocities in section 2.3. The observing point of José and María are examples

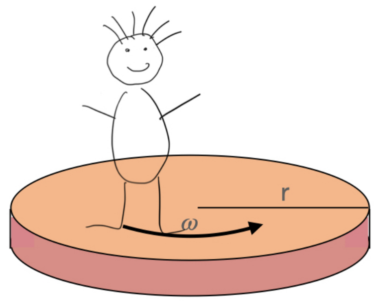


Figure 3: The length of the circumference of the rotating disk contracts for an observer external to it, but the radius does not.

of observers in inertial systems of reference, i.e. observers who were only differentiated from each other by a constant velocity. Ehrenfest experiment also involves two systems of reference. We can also think of María and José. But in this case José is standing on a spinning disk and María is watching him from a standing point outside the disk. The spinning disk is not an inertial system of reference. It is an accelerated one because all points on the disk, like the ones along the rim, are changing the direction of their velocity: they are experiencing a centripetal acceleration. But also a result from special relativity is that the times and lengths that observers in relative motion measure are different. María would perceive that the circumference of the disk is shorter than the one measured by José (see figure 3). And at the same time she would not perceive a change in the radius of the disk because, being perpendicular to the direction of motion it does not experience any shortening from her perspective. The effective result is that María would measure a circumference that is not equal to the number π multiplied by the disk diameter, which is the standard result from Euclidean geometry. But this means that the geometry involved is not Euclidean anymore but one where the space is not flat but "curved" or warped. The logical consequence of this paradox is that in accelerated systems of reference the geometry is not flat anymore. And due to the equivalence between accelerated systems and gravitation this meant that a gravitational field has to be necessarily associated with a non-flat geometry. This paradox suggested to Einstein that a new geometry was needed and that it would play a crucial role in his theory.

The theory of special relativity included new geometric ideas. Observers who carry out experiments in systems with relative motion measure different times and distances. The Lorentz transformations provide a way to consistently describe the motion of a given body from two different systems which are in motion with respect to each other. The fact that Lorentz transformations involved both space and time put them on an equal footing, without time having a preferred role. This led to the concept of a space-time continuum in four dimensions (three spatial ones plus time as a fourth dimension). This geometrical construction received the name of Minkowski space-time. But this new geometrical construction still had an Euclidean "flat" nature. To naturally incorporate gravity a different type of geometry, a "curved" geometry, was needed.