Lesson 15

Inflation

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So far our Big-Bang model is accepted to be radiation dominated at early times, matter dominated later on and most likely with a late transition to vacuum dominated.
This picture is good at describing many observations.
But: what are the initial conditions?
There are two aspects of the universe that point to a high degree of fine tuning: spatial flatness and the high degree of isotropy and homogeneity.
Could it be that a mechanism, starting with arbitrary initial conditions, turn out a smooth universe?

The flatness problem

Let's consider the Friedmann equation (18 in Lesson 13) in a universe with matter and radiation but no vacuum energy.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad (18)$$

which we will write

$$H^2 = \frac{1}{3\bar{m}_P^2}(\rho_M + \rho_R) - \frac{k}{a^2} \qquad (1)$$

In this equation the third term falls off as a^{-2} while $\rho_M \propto a^{-3}$ and $\rho_R \propto a^{-4}$. If we recall that we can write

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}}$$

where $\rho_{crit} = \frac{3H^2}{8\pi G} = 3\bar{m}_P^2 H^2$ the Friedmann eq (1) can be written

$$\Omega - 1 = \frac{k}{H^2 a^2} \qquad (2)$$

Then why is $\frac{k}{a^2}$ so close in order of magnitude to $\frac{\rho_M + \rho_R}{3\bar{m}_P^2}$.
This essentially means why $\Omega \approx 1$?

The horizon problem

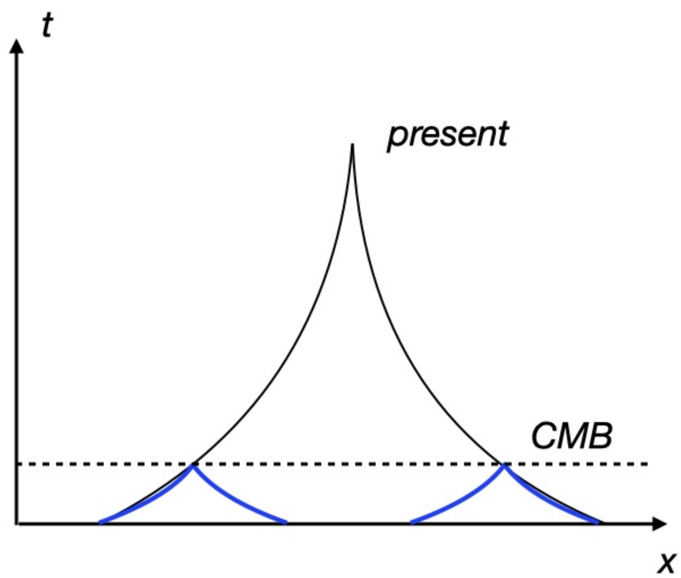

Fig 1 The horizon problem (adapted from Carroll).

Horizon exist because we are dealing with finite times and consequently finite distances as a result of the speed of light being a universal constant. Let's look at a photon moving along a radial trajectory in a flat universe.

$$0 = ds^2 = -dt^2 + a^2 dr^2. \tag{3}$$

The comoving (coordinate) distance traveled

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)} \tag{4}$$

to get the distance for any observer at a time t all we need is to multiply Δr by $a(t)$. If we assume, i.e. a matter dominated universe $a = \left(\frac{t}{t_0}\right)^{2/3}$. ($a_0$ is 1). The Hubble parameter is

$$H = \frac{2}{3} t^{-1} = a^{-3/2} H_0 \tag{5}$$

Then the photon travels a comoving distance

$$\Delta r = \frac{2}{H_0} \left(\sqrt{a_2} - \sqrt{a_1}\right). \tag{6}$$

The comoving horizon size at any value of $a = a_*$ is

$$r_{hor}(a_*) = \frac{2}{H_0} \sqrt{a_*}. \tag{7}$$

The physical horizon size as measure at the spatial hypersurface at a_* is

$$d_{hor}(a_*) = a_* r_{hor}(a_*) = 2H_*^{-1}. \tag{8}$$

Similarly had we assumed that the content was matter plus radiation in a nearly flat universe at any epoch we would've had

$$d_{hor}(a_*) \sim H_*^{-1} = d_H(a_*). \tag{9}$$

where the Hubble distance is

$$d_H = H^{-1}c. \tag{10}$$

Beware: Horizon distance is not the same concept as Hubble distance

The CMB is very isotropic. When we look at it we are observing the universe at a scale factor

$$a_{CMB} \approx 1/1200. \tag{11}$$

Remember that the redshift

$$1 + z = \frac{a_0}{a(t)}. \tag{12}$$

from (5) we have that the comoving distance between a point on the CMB and an observer on Earth is

$$\Delta r = 2H_0^{-1}(1 - \sqrt{a_{CMB}}) \approx 2H_0^{-1}. \tag{13}$$

but the comoving horizon distance is

$$r_{hor}(a_{CMB}) = 2H_0^{-1}\sqrt{a_{CMB}} \approx 6 \times 10^{-2} H_0^{-1}. \tag{14}$$

Conclusion: if we observe two widely-separated parts of the CMB, they will have no overlapping horizons. Distinct patches of the sky were causally disconnected at recombination.

A possible explanation is a period of **inflation**. An era in the early, very early universe when $\ddot{a} > 0$. If we find a mechanism that makes this possible then both the horizon and flatness problem are simultaneously solved.

Let's consider this inflationary phase as caused by the vacuum energy of the potential of a scalar field: the **inflaton.** From eq (76) slides Lesson 14,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{15}$$

and the Friedmann equation

$$H^2 = \frac{1}{3\bar{m}_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{16}$$

where we have taken $k = 0$ due to the flattening of the universe by inflation.

Inflation can occur if the evolution of the field is sufficiently gradual that the potential energy dominates the kinetic energy, and $\ddot{\phi}$ is small enough to allow this evolution to be maintained for a period sufficiently long.

So we need

$$\begin{aligned} \dot{\phi}^2 &\ll V(\phi), \\ |\ddot{\phi} &\ll |3H\dot{\phi}|, |V'|. \end{aligned} \tag{17}$$

To satisfy these conditions there are two parameters that need to be quite small:

they are known as **slow-roll parameters**:

$$\epsilon = \frac{1}{2}\bar{m}_P^2 \left(\frac{V'}{V}\right)^2,$$
$$\eta = \bar{m}_P^2 \left(\frac{V''}{V}\right) \tag{18}$$

We observe that while $\epsilon > 0$ that's not necessarily the case for η.
Consider now a simple example:

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{19}$$

With this potential

$$\epsilon = \eta = \frac{2\bar{m}_P^2}{\phi^2}. \tag{20}$$

For large ϕ we can get the slow-roll parameters as small as we want. But the energy density should not be as high as the Planck scale $\rightarrow \phi \ll \bar{m}_P^2/m$. If we start the field with a value ϕ_i, the number of e-folds before inflation ends (i.e. before the slow roll parameters become of order unity) is given by

$$N = \int_{t_i}^{t_e} H dt \approx -\frac{1}{\bar{m}_P^2} \int_{\phi_i}^{\phi_e} \frac{V}{V'} d\phi$$
$$\approx \frac{\phi_i^2}{4\bar{m}_P^2} - \frac{1}{2}. \tag{21}$$

To get a 60 e-fold increase we need $\phi_i > 16\bar{m}_P$. With the upper limit on the energy density there is an upper limit on the mass parameter $m \ll \bar{m}/16$. But there is no lower limit on m. Had we tried a $\lambda\phi^4$ potential we would've reached a similar conclusion, i.e. that λ needs to be quite small as compared with $\bar{m}_P$. That's the reason what it is said that the inflaton field has to be weakly coupled.

At some point inflation ends and the energy in the inflaton potential is converted into a thermalized gas of matter and radiation, a process called “reheating”. A good “understanding” of reheating is crucial. From it comes or disappear various relics of the process. i.e. reheating can inflate away relics that could have been produced in the early universe and we don’t observe now, like superheavy magnetic monopoles.

Inflation can dilute the monopole abundance appropriately, although they could be produced anew if the universe reheats to above the temperature of the gran-unification phase transition.

The Grand Unification (GUT) phase transition occurred roughly 10^{-36} to 10^{-35} seconds after the Big Bang, with an energy scale of approximately 10^{15} to 10^{16} GeV. The temperature of the universe during this transition was roughly 10^{27} to 10^{29} K, where strong and electroweak forces separated.

A crucial element of inflationary scenarios is the production of density perturbations which may be the origin of the CMB temperature anisotropies and the large scale structures in galaxies that we observe.

Inflation will attenuate any particle density quickly to zero, leaving behind only vacuum. But a vacuum state in an accelerating universe has a non zero temperature, the Gibbons-Hawking temperature. For a universe dominated by a potential energy V the Gibbons-Hawking temperature is

$$T_{GH} = \frac{H}{2\pi} \sim \frac{V^{1/2}}{\bar{m}_P}. \tag{22}$$

Corresponding to this temperature are the fluctuations in the inflaton field ϕ at each wavenumber k, with magnitude

$$|\Delta\phi|_k = T_{GH}. \tag{23}$$

We have assume a slow roll down which implies that the field is nearly flat so the fluctuations in ϕ lead

$$\delta\rho = V'(\phi)\delta\phi. \tag{24}$$

Inflation therefore produces $\delta\rho$ on every scale. The amplitude is nearly equal at each wavenumber. But there will be small variations due to the gradual change in V. The root-mean -square (RMS) density fluctuation is

$$\left.\frac{\delta\rho}{\rho}\right|_{rms} = \sqrt{\left\langle \left(\frac{\delta\rho}{\rho}\right)^2 \right\rangle}. \tag{25}$$

For statistically isotropic perturbations using Fourier analysis

$$\left(\left.\frac{\delta\rho}{\rho}\right|_{rms}\right)^2 = \int \Delta^2(k) d(\ln k), \tag{26}$$

where the dimensionless power spectrum

$$\Delta^2(k) \equiv \frac{k^3 |\delta_k|^2}{2\pi^2}, \tag{27}$$

and δ_k is the expectation value of the Fourier transform of the fractional density perturbation

$$\delta_k = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{k}\cdot\vec{x}} \frac{\delta\rho}{\rho} d^3x, \tag{28}$$

which we assume isotropic. The dimensionless power spectrum is a function of time. It is common to express the amplitude of the perturbations at the moment when the physical wavelength of the mode $\lambda = a/k$ is equal to the Hubble radius H^{-1}

$$A_S^2(k) \equiv \Delta^2(k)\Big|_{k=aH}, \tag{29}$$

$A_S(k)$ measures the amplitude for different modes at different times. For inflation driven by a slowly-rolling scalar field,

$$A_S^2(k) \sim \left.\frac{V^3}{\bar{m}_P^6 (V')^2}\right|_{k=aH} \sim \left.\frac{V}{\bar{m}_P^4 \epsilon}\right|_{k=aH}, \quad (30)$$

The s in the spectrum because it describes scalar fluctuations in the metric. These are tied to the energy-momentum distribution, and the density fluctuations produced by inflation are adiabatic. The fluctuations are also Gaussian: the phases of the Fourier modes describing fluctuations at different scales are uncorrelated. These aspects of inflation, a nearly scale free spectrum of adiabatic density fluctuations with a Gaussian distribution are consistent with current observations of the CMB and large scale structure.

Not only the nearly massless inflaton is excited during inflation, but also other nearly massless particle. For example the graviton (tensor perturbations in the metric). This propagates excitations of the gravitational field. Tensor fluctuations have a spectrum

$$A_T^2(k) \sim \left. \frac{V}{\bar{m}_P^4} \right|_{k=aH}, \tag{31}$$

This depends only on the potential and not on the derivatives. If we were to observe them, they would give direct information about the energy scale of inflation. To parametrize both the scalar and tensorial spectra

$$A_S^2(k) \propto k^{n_s - 1},$$

and

$$A_T^2(k) \propto k^{n_T}, \tag{32}$$

where n_S and n_T are the spectral indices. They are related to the slow-roll parameters of the potential by

$$n_S = 1 - 6\epsilon + 2\eta,$$

and

$$n_T = -2\epsilon. \tag{33}$$

A consistent relationship between the perturbations and the amplitudes involved is given by

$$\frac{(\Delta T/T)^2_T}{(\Delta T/T)^2_S} = -7 n_T. \tag{34}$$

The existence of tensor perturbations is a crucial prediction of inflation. It could be verified through observations of the polarization of the CMB. Although there are two types of polarizations, the electric E-modes (curl-free part) are generated by scalar perturbations while the tensor ones are generated by the curl part (B-modes). Of course it will require not only detecting the polarization but also measuring the spectral index of it. Tensor perturbations depend only on V, not on its derivatives (there are fluctuations of the metric itself). If the CMB anisotropies seen by COBE are due to tensor fluctuations

$$V_{inflation} \sim (10^{16}\mathrm{GeV})^4 \qquad (35)$$

(constrained by 60 fold increase). Even if there were scalar only in nature

$$V_{inflation}^{1/4} \sim \epsilon^{1/4} 10^{16}\mathrm{GeV} \tag{36}$$