Introduction to General Relativity 2026
Lesson 2: A primer on Tensor Algebra

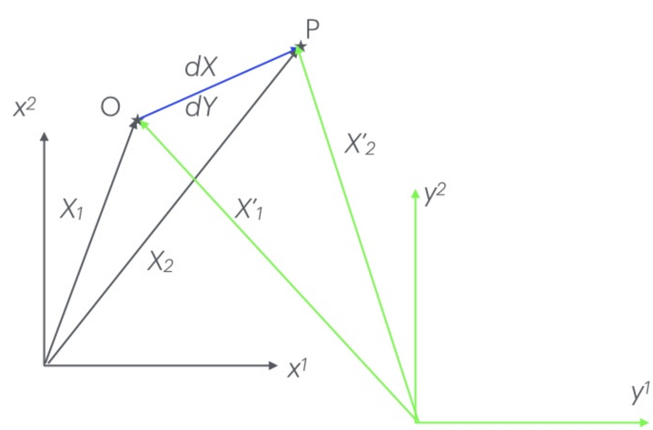
Mario C Díaz

Glossary and keywords

Transformation of coordinates. Contravariant and covariant transformations. One-forms. Einstein's convention. Operations with tensors.

Contravariant vectors

What is a vector? How do vectors transform? A vector indicates a point in space which exists independently of our description of it. We need to use a reference system to describe it. But when doing so we make clear information about that point and allow other observers to describe the same point from their own reference systems. A vector could be also a function of time, indicating the displacement of an object in space. Similarly the trajectory of an object in space is independent of us, but we need a reference system to describe it.



A displacement vector from two different coordinate systems.

The question arises then: how are the coordinates of this displacement in the two systems related? If we think that the displacement is an infinitesimal one then the coordinates of it in the system X

are:

$$dX^i = [dX^1, dX^2, ..., dX^n] \tag{1}$$

where n is the dimension of our space. If we look at the same displacement described from the Y system we have:

$$dY^i = [dY^1, dY^2, ..., dY^n] \tag{2}$$

We learn from calculus that these coordinates are related by

$$dY^i = \sum_j \frac{\partial Y^i}{\partial X^j} dY^j \tag{3}$$

We will follow Einstein's convention and eliminate the $\sum$ symbol when indices are repeated. We understand that repeated indices indicate a sum over them.

$$dY^i = \frac{\partial Y^i}{\partial X^j} dY^j \tag{4}$$

Notice that, i.e. in two dimensions (3) or (4) can be written as vector equation in this manner:

$$\begin{pmatrix} dY^1 \\ dY^2 \end{pmatrix} = \begin{pmatrix} \partial Y^1/\partial X^1 & \partial Y^1/\partial X^2 \\ \partial Y^2/\partial X^1 & \partial Y^2/\partial X^2 \end{pmatrix} \begin{pmatrix} dX^1 \\ dX^2 \end{pmatrix} \tag{5}$$

$||\partial Y^i/\partial X^j||$ is the matrix of the transformation of coordinates. Equation (5) is the definition of how a contravariant vector transforms. We can define a vector as a quantity V described in a coordinate system X^i that when a different coordinate system X'^i is used to describe it, it transforms:

$$(V')^i = \frac{\partial (X')^i}{\partial X^j} V^j \tag{6}$$

These vectors are called contravariant vectors.

Covariant vectors

Let's examine now another type of vector. The gradient of a scalar function is a vector but we will see that it transforms differently from the way contravariant vectors like the displacement vector transform. If we have a scalar function of the coordinates $\Phi(X^i)$ the gradient will be:

$$\vec{W} = \nabla\Phi = [\partial\Phi/\partial x^1, \partial\Phi/\partial x^2, ..., \partial\Phi/\partial x^n] \tag{7}$$

where n is the dimension of our space. How do the coordinates of the gradient transform under a coordinate transformation to a system $(X')^i$? The calculus chain rule shows it clearly:

$$\frac{\partial\Phi}{\partial (x')^i} = \frac{\partial\Phi}{\partial x^j} \frac{\partial x^j}{\partial (x')^i} \tag{8}$$

or

$$(W')_i = W_j \frac{\partial x^j}{\partial (x')^i} \tag{9}$$

remember that there is an implicit sum in the repeated indices. A covariant vector transforms with the inverse of the transformation matrix. Contravariant indicates that if we change the unit vectors by a factor α the components of the vector in that base will changed by a factor $1/\alpha$. In the case of a covariant vector its components will change just in the same factor α, *co-variantly*.

Vectors, Tensors and Transformations

Example 1
Find the components of the tangent vector to the curve consisting of a circle of radius a centered at the origin in Cartesian and polar coordinates:

Solution

$(x^a) \to (x'^a)$ where x^a are Cartesian coordinates gives x'^a

$$R = (x^2 + y^2)^{1/2}$$
$$\phi = \tan^{-1}(y/x)$$

Then the matrix of coordinate transformations is

$$\left(\frac{\partial x'^a}{\partial x^b}\right) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi/R & \cos\phi/R \end{pmatrix}$$

and

$$X^a = \frac{dx^a}{d\phi} = (-a\sin\phi, a\cos\phi),$$

$$\begin{aligned} X'^a =& \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi/R & \cos\phi/R \end{pmatrix}\Bigg|_{R=a} \cdot \begin{pmatrix} -a\sin\phi \\ a\cos\phi \end{pmatrix} \\ X'^a =&(0,1) \end{aligned}$$

What about $\frac{dx^a}{dR}$?

$$X^a = \frac{dx^a}{dR} = (\cos\phi, \sin\phi),$$

and

$$
\begin{aligned}
X'^a =& \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi/R & \cos\phi/R \end{pmatrix} \cdot \begin{pmatrix} \cos\phi, \\ \sin\phi \end{pmatrix} \\
=& X'^a = (1, 0)
\end{aligned}
$$

Example 2

Let now the (x^a) be Cartesian coordinates and (x'^a) plane polar ones.

a) Find X'^a if $X^a = (1, 0)$.

$$
\begin{aligned}
X'^a =& \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi/R & \cos\phi/R \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
X'^a =& (\cos\phi, -\sin\phi/R)
\end{aligned}
\tag{10}
$$

b) Find expressions for $\partial/\partial x, \partial/\partial y, \partial/\partial R, \partial/\partial\phi$

We remember that basis vectors transform with the inverse of the transformation that relates vectors in two coordinate systems. If we want to get $(\partial/\partial x, \partial/\partial y)$ then:

$$
\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi/R \\ sin\phi & \cos\phi/R \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial R \\ \partial/\partial\phi \end{pmatrix}
$$

And we get:

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \cos\phi\frac{\partial}{\partial R} - \frac{\sin\phi}{R}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial y} &= \sin\phi\frac{\partial}{\partial R} + \frac{\cos\phi}{R}\frac{\partial}{\partial\phi}, \\
\frac{\partial}{\partial R} &= \frac{x}{(x^2+y^2)^{1/2}}\frac{\partial}{\partial x} + \frac{y}{(x^2+y^2)^{1/2}}\frac{\partial}{\partial y} \\
\frac{\partial}{\partial\phi} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}
\end{aligned}
$$

c) Express the vector field $\vec{A} = (1, 0)$ in terms of the basis (i.e. as an operator):

$$
X^a\partial_a = \frac{\partial}{\partial x}
$$

$$
X'^a\partial'_a = \cos\phi\frac{\partial}{\partial R} - \frac{\sin\phi}{R}\frac{\partial}{\partial\phi}
$$

d) If $Y^a = (0,1)$ and $Z^a = (-y, x)$ find Y'^a, Z'^a:

$$Y'^a = (\sin\phi, \cos\phi/R),$$
$$Z'^a = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi/R & \cos\phi/R \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \end{pmatrix}$$

And of course the vectors in the two systems are

$$Y = \frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial R} + \frac{\cos\phi}{R}\frac{\partial}{\partial\phi},$$
$$Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial\phi}$$

The scalar product

Let's consider the representation of two vectors $\vec{A}$ and $\vec{B}$ on the vector basis in some frame $\mathcal{O}$:

$$\vec{A} = A^\alpha \vec{e}_\alpha, \qquad \vec{B} = B^\beta \vec{e}_\beta$$

The scalar product is:

$$\vec{A} \cdot \vec{B} = (A^\alpha \vec{e}_\alpha) \cdot (B^\beta \vec{e}_\beta)$$

which can be put:

$$\vec{A} \cdot \vec{B} = A^\alpha B^\beta (\vec{e}_\alpha \cdot \vec{e}_\beta), \tag{11}$$

We define the metric tensor:

$$\vec{A} \cdot \vec{B} = A^\alpha B^\beta \eta_{\alpha\beta}, \tag{12}$$

The numbers $\eta_{\alpha\beta}$ are called the components of the metric tensor. Comments:

1. metric means that the quantity is associated with giving a ”notion” of measure (the ”length” of a scalar product).

2. tensor means that it is a quantity with special transformation properties.

3. it is important to understand the ”meaning” of the double summation.

One way of understanding it in the language of matrices is seeing this:

$$\vec{A} \cdot \vec{B} = \begin{pmatrix} A^0 & A^1 & A^2 & A^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} \tag{13}$$

When performing the double summation you have to be careful, i.e.:

$$\vec{A} \cdot \vec{B} = A^0 B^\beta \eta_{0\beta} + A^1 B^\beta \eta_{1\beta} + A^2 B^\beta \eta_{2\beta} + A^3 B^\beta \eta_{3\beta}$$

and for each repeated β subindex there is a sum to do where β also runs from 0 to 3.

Definition of tensors

A tensor of type $\begin{pmatrix} 0 \\ N \end{pmatrix}$ is a function of *N* vectors, which is linear in each of its *N* arguments. i.e. **g** is a tensor of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

$$\mathbf{g}(\vec{A}, \vec{B}) := \vec{A} \cdot \vec{B} \tag{14}$$

and

$$\mathbf{g}(\alpha\vec{A} + \beta\vec{B}, \vec{C}) = \alpha\mathbf{g}(\vec{A}, \vec{C}) + \beta\mathbf{g}(\vec{B}, \vec{C}) \tag{15}$$

A function is a tensor of type $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensors: one-forms

These tensors are generally called covectors. One forms are denoted $\widetilde{p}$ to differentiate them from vectors like $\vec{A}$. A little bit more rigorous definition of one forms would be: they are linear, real valued functions of vectors. A one form $\widetilde{\omega}$ at a point P in a given space (manifold) associates with a vector $\vec{V}$ at P (notice this P again) a real number, we call $\widetilde{\omega}(\vec{V})$.

Forms follow these rules:

$$\widetilde{s} = \widetilde{p} + \widetilde{q}, \tag{16}$$

$$\widetilde{r} = \alpha\widetilde{p}, \tag{17}$$

$$\widetilde{s}(\vec{A}) = \widetilde{p}(\vec{A}) + \widetilde{q}(\vec{A}), \tag{18}$$

$$\widetilde{r}(\vec{A}) = \alpha\widetilde{p}(\vec{A}), \tag{19}$$

The set of one forms satisfies the axioms for a vector space.
These axioms are 1) Associativity of addition, 2) Commutativity of addition, 3) Identity element of addition, 4) Inverse elements of addition 5) Distributivity of scalar multiplication with respect to vector addition, 6) Distributivity of scalar multiplication with respect to field addition, 7) Compatibility of scalar multiplication with field multiplication, and 8) Identity element of scalar multiplication.

Components of $\widetilde{p}$ are:

$$\widetilde{p}_\alpha := \widetilde{p}(\vec{e}_\alpha), \tag{20}$$

$$\widetilde{p}(\vec{A}) = \widetilde{p}(A^\alpha \vec{e}_\alpha), \; = A^\alpha \widetilde{p}(\vec{e}_\alpha) \tag{21}$$

$$\widetilde{p}(\vec{A}) = A^\alpha \widetilde{p}_\alpha \tag{22}$$

Notice that contraction does not involve any other tensor or rule, i.e.

$$\widetilde{p}(\vec{A}) = A^0 p_0 + A^1 p_1 + A^2 p_2 + A^3 p_3$$

while two vectors can not make a scalar without the help of the metric tensor.

Example:
The row vector $(-4, 9, 1)$. Why?

$$\begin{pmatrix} -4, & 9, & 1 \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -4, & 9, & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= -4x + 9y + z,$$

What are the components of $\widetilde{p}$ on a basis $\{\vec{e}_{\bar{\beta}}\}$

$$p_{\bar{\beta}} := \widetilde{p}(\vec{e}_{\bar{\beta}}) = \widetilde{p}(\Lambda^\alpha{}_{\bar{\beta}} \vec{e}_\alpha) = \Lambda^\alpha{}_{\bar{\beta}}\, \widetilde{p}(\vec{e}_\alpha)$$
$$= \Lambda^\alpha{}_{\bar{\beta}}\, \widetilde{p}_\alpha$$

Components of one-forms transform as basis vectors do and opposite to component of vectors (using the inverse transformation).

This means that $A^\alpha p_\alpha$ is frame independent. (Exercise: prove it).

Basis of one-forms
We want a set of $\{\widetilde{\omega}^\alpha, \alpha = 0, ..., 3\}$ "dual" to $\{\vec{e}_\alpha\}$ such that,

$$\widetilde{p} = p_\alpha \widetilde{\omega}^\alpha. \tag{23}$$

Remembering eq (12) we see from (13) that:

$$\begin{aligned} \widetilde{p}(\vec{A}) = p_\alpha \widetilde{\omega}^\alpha(\vec{A}) & \\ &= p_\alpha \widetilde{\omega}^\alpha (A^\beta \vec{e}_\beta) \\ &= p_\alpha A^\beta \widetilde{\omega}^\alpha (\vec{e}_\beta). \end{aligned}$$

This can only be equal to $p_\alpha A^\alpha$ if

$$\widetilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta \tag{24}$$

These components can be written:

$$\widetilde{\omega}^0 \underset{\mathcal{O}}{\longrightarrow} \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array} \right) \tag{25}$$

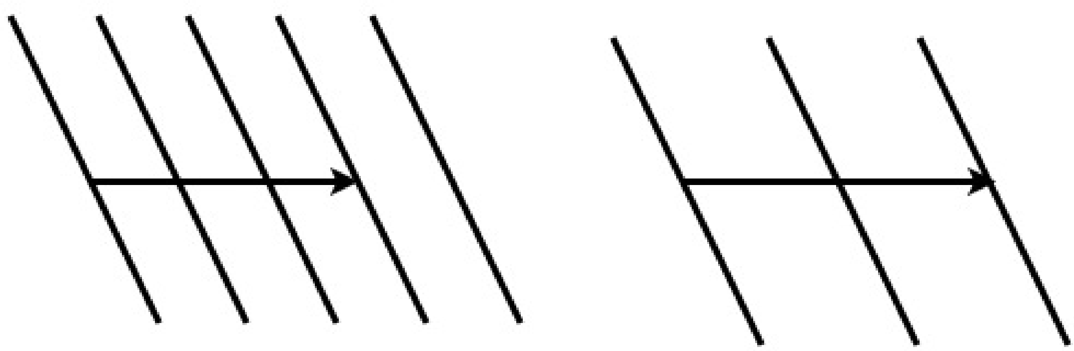
$$\widetilde{\omega}^1 \underset{\mathcal{O}}{\longrightarrow} \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array} \right) \tag{26}$$

$$\widetilde{\omega}^2 \underset{\mathcal{O}}{\longrightarrow} \left(\begin{array}{cccc} 0 & 0 & 1 & 0 \end{array} \right) \tag{27}$$

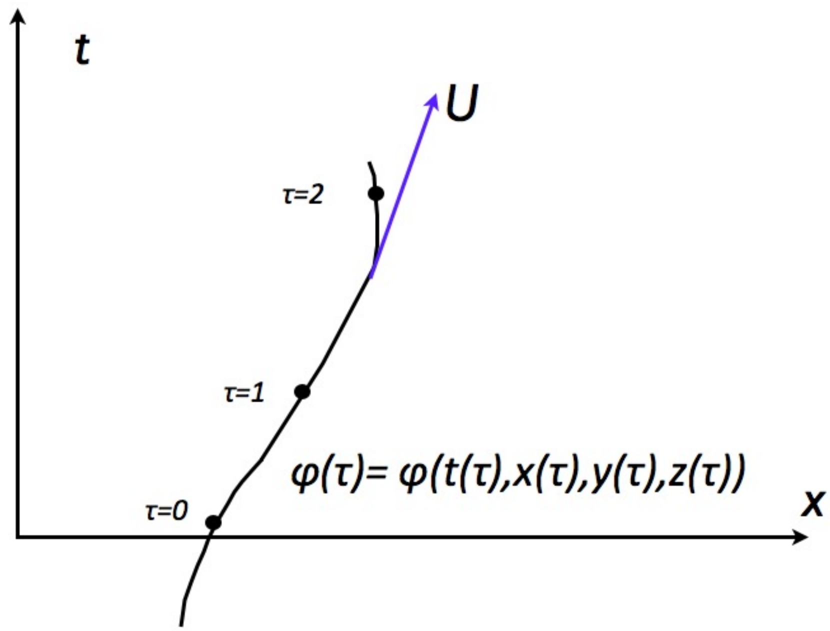
$$\widetilde{\omega}^3 \underset{\mathcal{O}}{\longrightarrow} \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} \right) \tag{28}$$

And

$$\widetilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta}\widetilde{\omega}^{\beta} \tag{29}$$



The one form can be seen as a series of surfaces.The value of a 1-form on a vector can be pictured as the number of surfaces the arrow perforates.



$\frac{d\phi}{d\tau} = \widetilde{d}\phi\vec{U}$ with $\widetilde{d}\phi \underset{\mathcal{O}}{\rightarrow} (\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$

We can see that $\widetilde{d}\phi$ is a consistent definition with the way one-forms transform:

$$\widetilde{d}\phi_{\bar{\alpha}} = \Lambda^{\beta}{}_{\alpha}\widetilde{d}\phi_{\beta} \tag{30}$$

$$\frac{\partial\phi}{\partial x^{\bar{\alpha}}} = \frac{\partial\phi}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} \tag{31}$$

(21) means $\widetilde{d}\phi_{\bar{\alpha}} = \frac{\partial x^{\beta}}{\partial x^{\alpha}}\widetilde{d}\phi_{\beta}$, but $x^{\beta} = \Lambda^{\beta}{}_{\bar{\alpha}}x^{\bar{\alpha}}$, so

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} = \Lambda^{\beta}{}_{\bar{\alpha}} \tag{32}$$

This is a fundamental identity and shows that the gradients transform with the inverse.

Derivatives and notation

$$\frac{\partial\phi}{\partial x} := \phi_{,x}$$

In general

$$\frac{\partial\phi}{\partial x^{\alpha}} := \phi_{,\alpha} \tag{33}$$

Notice that:

$$x^{\alpha}_{,\beta} \equiv \delta^{\alpha}{}_{\beta},$$

If we think of a generic 1-form basis $\widetilde{d}x^\alpha$ we can remember of:

$$\widetilde{d}f = \frac{\partial f}{\partial x^\alpha}\widetilde{d}x^\alpha$$

which is showing an "operator" of the form:

$$\widetilde{d} = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \begin{pmatrix} \widetilde{d}t \\ \widetilde{d}x \\ \widetilde{d}y \\ \widetilde{d}z \end{pmatrix}$$

This suggests that:

$$\widetilde{d}x^\alpha := \widetilde{\omega}^\alpha \tag{34}$$

Normal one-forms

$\widetilde{d}f$ is normal to surfaces of constant f.

$$\widetilde{d}f = 0 = (\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})(\widetilde{d}t, \widetilde{d}x, \widetilde{d}y, \widetilde{d}z)$$

Which means that $\widetilde{d}x^\alpha$ is orthogonal to the vectors tangent to the surface at the points where $f(\vec{x})$ is constant.

The $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors

$$f_{\alpha\beta} := \mathbf{f}(\vec{e}_\alpha, \vec{e}_\beta) \tag{35}$$

Once we know these values it is easy to find them for arbitrary vectors:

$$\begin{aligned} \mathbf{f}(\vec{A}, \vec{B}) = \mathbf{f}(A^\alpha \vec{e}_\alpha, B^\beta \vec{e}_\beta) \\ &= A^\alpha B^\beta \mathbf{f}(\vec{e}_\alpha, \vec{e}_\beta) \\ &= A^\alpha B^\beta f_{\alpha\beta} \end{aligned}$$

Can we write it in terms of "two forms" basis?, i.e. $\mathbf{f} = f_{\alpha\beta}\widetilde{\omega}^{\alpha\beta}$ for some $\widetilde{\omega}^{\alpha\beta}$

But then

$$f_{\mu\nu} = \mathbf{f}(\vec{e}_\mu, \vec{e}_\nu) = f_{\alpha\beta}\widetilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu)$$

and this would imply:

$$\widetilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu) = \delta^\alpha{}_\mu \delta^\beta{}_\nu \tag{36}$$

But from what we have learned before:

$$\widetilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu) = \widetilde{\omega}^\alpha(\vec{e}_\mu)\widetilde{\omega}^\beta(\vec{e}_\nu)$$

So we have something special as a basis:

$$\widetilde{\omega}^{\alpha\beta} = \widetilde{\omega}^\alpha \bigotimes \widetilde{\omega}^\beta \tag{37}$$

$$\mathbf{f} = f_{\alpha\beta}\widetilde{\omega}^\alpha \bigotimes \widetilde{\omega}^\beta \tag{38}$$

Symmetries

We will say that $\mathbf{f}$ is symmetric if:

$$\mathbf{f}(\vec{A}, \vec{B}) = \mathbf{f}(\vec{B}, \vec{A}) \qquad \forall \vec{A}, \vec{B}, \tag{39}$$

which implies:

$$f_{\alpha\beta} = f_{\beta\alpha} \tag{40}$$

An arbitrary $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor $\mathbf{h}$ can have associated a symmetric $\mathbf{h}_{(s)}$:

$$\mathbf{h}_{(s)}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) + \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}). \tag{41}$$

which in components:

$$h_{(s)_{\alpha\beta}} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha})$$

And there is a special notation for it:

$$h_{(\alpha\beta)} := \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) \tag{42}$$

We will say that $\mathbf{f}$ is antisymmetric if:

$$\mathbf{f}(\vec{A}, \vec{B}) = -\mathbf{f}(\vec{B}, \vec{A}) \qquad \forall \vec{A}, \vec{B}, \tag{43}$$

which implies:

$$f_{\alpha\beta} = -f_{\beta\alpha} \tag{44}$$

An arbitrary $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor $\mathbf{h}$ can have associated a antisymmetric $\mathbf{h}_{(A)}$:

$$\mathbf{h}_{(A)}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) - \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}). \tag{45}$$

which in components:

$$h_{(A)_{\alpha\beta}} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha})$$

And there is a special notation for it:

$$h_{[\alpha\beta]} := \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) \tag{46}$$

Notice this important result:

$$h_{\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) + \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) = h_{(\alpha\beta)} + h_{[\alpha\beta]} \tag{47}$$

The metric tensor $\mathbf{g}$ is so far symmetric:

$$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}(\vec{B}, \vec{A}) \tag{48}$$

Mapping vectors into one-forms

Each vector has its soulmate, i.e. a one-form associated. The metric does the"mapping".

$$\mathbf{g}(\vec{V}, \quad) := \widetilde{V}(\quad), \tag{49}$$

$$\widetilde{V}(\vec{A}) := \mathbf{g}(\vec{V}, \vec{A}) = \vec{V} \cdot \vec{A}. \tag{50}$$

What is $\widetilde{V}$?, or what are its components?

$$\begin{aligned} V_\alpha := \widetilde{V}(\vec{e}_\alpha) \\ &= \vec{V} \cdot \vec{e}_\alpha = \vec{e}_\alpha \cdot \vec{V} \\ &= \vec{e}_\alpha \cdot (V^\beta \vec{e}_\beta) \\ &= (\vec{e}_\alpha \cdot \vec{e}_\beta) V^\beta \end{aligned}$$

But we know what $(\vec{e}_\alpha \cdot \vec{e}_\beta)$ is:

$$V_\alpha = \eta_{\alpha\beta} V^\beta \tag{51}$$

Example

$$\begin{aligned} V_0 = V^\beta \eta_{\beta 0} \\ &= V^0 \eta_{00} + V^1 \eta_{10} + \cdots \\ &= V^0(-1) + 0 + 0 + 0 \\ &= -V^0 \end{aligned}$$

$$\begin{aligned} V_1 = V^\beta \eta_{\beta 1} \\ &= V^0 \eta_{01} + V^1 \eta_{11} + \cdots \\ &= +V^1 \end{aligned}$$

The inverse mapping

There is an inverse to $\eta_{\alpha\beta}$ which we call $\eta^{\alpha\beta}$. And we will use it to find A^α when given A_α.

$$A^\alpha := \eta^{\alpha\beta} A_\beta \tag{52}$$

which is consistent with:

$$A_\beta = \eta_{\beta\alpha} A^\alpha \tag{53}$$

$\eta^{\alpha\beta}$ is uniquely defined by:

$$\eta^{\beta\alpha} \eta_{\alpha\gamma} = \delta^\beta{}_\gamma \tag{54}$$

Exercise: find η^{00}

Why all this fuss about forms and vectors?

The duality is manifest when we deal with a metric space. The metric makes vectors and one forms different. In Euclidean space duality is meaningless, because one-forms and vectors would have the same components. Rigorous mathematical definition for dual space: Any vector space, V, has a corresponding dual vector space (or just dual space) consisting of all linear functionals on V.

In special relativity it works the way we just learned it. The concept is useful in many other areas of physics, including Quantum Mechanics and Solid State physics. The Fourier transform can be formulated mathematically to provide a dual representation of a function of time.

Magnitudes of one-forms

$$\widetilde{p}^2 = \vec{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta \tag{55}$$

Of course:

$$\widetilde{p}^2 = \eta^{\alpha\beta} p_\alpha p_\beta \tag{56}$$

They can be as vectors spacelike, timelike or null.

The $\begin{pmatrix} M \\ N \end{pmatrix}$ tensors

Vectors are not really privileged. They can be seen are linear functions of one-forms into real numbers.

$$\vec{V}(\widetilde{p}) \equiv \widetilde{p}(\vec{V}) \equiv p_\alpha V^\alpha \equiv \langle \widetilde{p}, \vec{V} \rangle \tag{57}$$

$\begin{pmatrix} M \\ 0 \end{pmatrix}$ tensors are linear functions of M one-forms into real numbers.

A simple $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is $\vec{V} \otimes \vec{W}$. Acting on forms would give: $\vec{V}(\widetilde{p})\vec{W}(\widetilde{q}) = V^\alpha p_\alpha W^\beta q_\beta$. A basis for $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensors is $\vec{e}_\alpha \otimes \vec{e}_\beta$

NOTE: $\begin{pmatrix} M \\ 0 \end{pmatrix}$ tensors have their indices as superscripts.

$\begin{pmatrix} M \\ N \end{pmatrix}$ **tensors**

An $\begin{pmatrix} M \\ N \end{pmatrix}$ tensor is a linear function of *M* one-forms and *N* vectors into real numbers. An $\begin{pmatrix} M \\ N \end{pmatrix}$ tensor will have *M* indices up and *N* indices down.

up $\rightarrow$ contravariant, down $\rightarrow$ covariant.

Raising and lowering indices

The metric maps a vector $\vec{V}$ into a one-form $\widetilde{V}$. It also generalizes this into the following: a metric maps an $\begin{pmatrix} N \\ M \end{pmatrix}$ tensor into an $\begin{pmatrix} N-1 \\ M+1 \end{pmatrix}$ tensor. Similarly the inverse maps an $\begin{pmatrix} N \\ M \end{pmatrix}$ tensor into and a $\begin{pmatrix} N+1 \\ M-1 \end{pmatrix}$ tensor.

For example a $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ tensor is mapped into a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor:

$$T^{\alpha}{}_{\beta\gamma} := \eta_{\beta\mu} T^{\alpha\mu}{}_{\gamma} \tag{58}$$

Or

$$T_{\alpha}{}^{\beta}{}_{\gamma} := \eta_{\alpha\mu} T^{\mu\beta}{}_{\gamma} \tag{59}$$

$$T^{\alpha\beta\gamma} := \eta^{\gamma\mu} T^{\alpha\beta}{}_{\mu} \tag{60}$$

Also (important)

$$\eta^{\alpha}{}_{\beta} \equiv \eta^{\alpha\mu} \eta_{\mu\beta} \tag{61}$$

Differentiation of tensors

$$\mathbf{T} = T^{\alpha}{}_{\beta} \widetilde{\omega}^{\beta} \otimes \vec{e}_{\alpha}$$

$$\frac{d\mathbf{T}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\mathbf{T}(\tau + \Delta\tau) - \mathbf{T}(\tau)}{\Delta\tau} \tag{62}$$

$$\frac{d\mathbf{T}}{d\tau} = (\frac{dT^{\alpha}{}_{\beta}}{d\tau}) \widetilde{\omega}^{\beta} \bigotimes \vec{e}_{\alpha} \tag{63}$$

where $dT^{\alpha}{}_{\beta}/d\tau = T^{\alpha}{}_{\beta,\gamma}U^{\gamma}$ Then

$$d\mathbf{T}/d\tau = (T^{\alpha}{}_{\beta,\gamma}\widetilde{\omega}^{\beta} \bigotimes \vec{e}_{\alpha})U^{\gamma} \tag{64}$$

from where we get:

$$\nabla\mathbf{T} := (T^{\alpha}{}_{\beta,\gamma}\widetilde{\omega}^{\beta} \bigotimes \widetilde{\omega}^{\gamma} \bigotimes \vec{e}_{\alpha}) \tag{65}$$

This tensor is called the gradient of $\mathbf{T}$ and $d\mathbf{T}/d\tau = \nabla_{\vec{U}}\mathbf{T}$ where

$$\nabla_{\vec{U}}\mathbf{T} \rightarrow \{T^{\alpha}{}_{\beta,\gamma}U^{\gamma}\} \tag{66}$$