Introduction to General Relativity 2026
Lesson 3: Physical Tensors as sources: matter-energy, electromagnetic fields and perfect fluids

Mario C Díaz

1 The Electromagnetic field strength tensor

The electromagnetic field strength tensor can be written:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}$$

Maxwell's equations

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J} \tag{1}$$
$$\nabla \cdot \mathbf{E} = \rho \tag{2}$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{3}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

In component notation:

$$\epsilon^{ijk} \partial_j B_k - \partial_0 E^i = J^i \tag{5}$$
$$\partial_i E^i = J^0 \tag{6}$$
$$\epsilon^{ijk} \partial_j E_k + \partial_0 B^i = 0 \tag{7}$$
$$\partial_i B^i = 0 \tag{8}$$

Notice that J^0 is the charge density. So $J^\mu = (\rho, J^x, J^y, J^z)$. Also noticing that $F^{0i} = E^i$ and $F^{ij} = \epsilon^{ijk} B_k$ Exercise: Using the definitions prove this is true. But this means that we can write (5) and (6) as:

$$\partial_j F^{ij} - \partial_0 F^{0i} = J^i \tag{9}$$
$$\partial_i F^{0i} = J^0 \tag{10}$$

and then combined:

$$\partial_\mu F^{\nu\mu} = J^\nu \tag{11}$$

And (7) and (8):

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \tag{12}$$

Exercise: Prove (11) and (12).
Hint: Use that

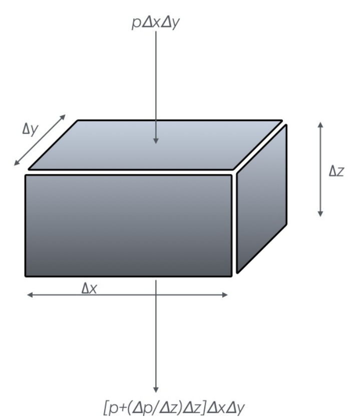
$$\partial_{[\mu} F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0 \tag{13}$$

2 Momentum and energy in a fluid

2.1 Perfect fluids and the Euler equation of Hydrodynamics

The physics of the Classical Mechanics of the XVIII century is an analysis of the kinematics and dynamics of point like particles. The XIX Century succeeds at treating the physics of gases as a collection of particles through the development of thermodynamics. Following on it is the treatment of fluids in a classical context. A model similar to the ideal gas is the concept of ideal fluids. What is an ideal fluid? An ideal fluid is an ensemble of a "zillion" of point like particles that is incompressible (you can not decrease its volume by squeezing it) and shows no internal resistance to flow (zero viscosity). We also assume that for an ideal fluid its particles do not rotate. Let's attempt to build an equation of motion for such a fluid within the framework of Newtonian Mechanics: let's consider an infinitesimal fluid element like the one depicted in figure. According to Newton's second law, the mass of such a fluid element times its acceleration should equal the net force acting on that fluid element.

If we take an element of unit volume, then we have



A fluid element undergoing pressure in all directions will satisfy

$$\rho \vec{a} = \vec{f}. \tag{14}$$

where $\vec{f}$ is the force per unit volume acting on the element and $\vec{a}$ is the acceleration of the fluid. This force may originate in different manners. Viscosity would constitute an effective force dissipating energy through friction. But we already decided to ignore it in the case of a perfect fluid. Gravity or Electromagnetic forces could also be acting due to the presence of external sources and depending on the existence of electric charges associated with the fluid. We would ignore external sources for now. The main force we like to consider is the one due to pressure gradients within the fluid. We can analyze its behavior in this manner:

Following the figure we can observe that in the z direction there is at the top of our infinitesimal element a pressure $p\Delta x\Delta y$ while at the bottom the value is $[p+(\Delta p/\Delta z)\Delta z]\Delta x\Delta y = [p+(\partial p/\partial z)\Delta z]\Delta x\Delta y$.

The difference is $p\Delta x\Delta y - p\Delta x\Delta y - (\partial p/\partial z)\Delta z\Delta x\Delta y = -(\partial p/\partial z)\Delta z\Delta x\Delta y$.
We can repeat the calculation in the other two directions and the result is

$$\rho a^i = f^i = -\frac{\partial p}{\partial x^i}. \tag{15}$$

or in vectorial notation

$$\rho\vec{a} = -\nabla p \tag{16}$$

Now we need to properly calculate the acceleration of the particles in the fluid. This is a crucial point in Hydrodynamics. Even if we have a steady flow field for a fluid, it may still experience acceleration due to moving to a position where the velocity field has a different value.

Precisely due to this fact is that the acceleration of an element of the fluid at point $\vec{x}$ at time t is not just $\partial\vec{v}/\partial t$. The reason is that the element is at that position only instantaneously. So the way to find the correct expression for the acceleration is to realize that an element at a position $\vec{x}$ at time t will be at a position $\vec{x}+\vec{v}\delta t$ at time $t+\delta t$. Consequently the change in its velocity in the interval δt is

$$\vec{v}(\vec{x}+\vec{v}\delta t) - \vec{v}(\vec{x},t) = \frac{\partial\vec{v}}{\partial x}v_x\delta t + \frac{\partial\vec{v}}{\partial y}v_y\delta t + \frac{\partial\vec{v}}{\partial z}v_z\delta t + \frac{\partial\vec{v}}{\partial t}\delta t \tag{17}$$

which dividing by δt and taking the limit for δt going to 0 gives

$$\lim_{\delta t\to 0}\frac{\vec{v}(\vec{x}+\vec{v}\delta t) - \vec{v}(\vec{x},t)}{\delta t} = \vec{a} \tag{18}$$

and then

$$\vec{a} = \frac{\partial\vec{v}}{\partial x}v_x + \frac{\partial\vec{v}}{\partial y}v_y + \frac{\partial\vec{v}}{\partial z}v_z + \frac{\partial\vec{v}}{\partial t} \equiv \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} \tag{19}$$

We now have an equation of motion for our fluid (notice that even if at a given element $\partial\vec{v}/\partial t =$) we still have acceleration. The resulting equation is called the Euler equation:

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\,\vec{v}\right]=-\nabla p \tag{20}$$

This equation and the equation of continuity are the equations of motion describing the flow of a non-viscous fluid.

2.2 The energy momentum tensor of a fluid

We learned with Special Relativity that we needed a 4-dimensional formalism to develop a covariant formalism of Classical Mechanics invariant under Lorentz transformations. In this manner we used a 4-dimensional momentum which incorporates the energy density as a time component added to the traditional space components.

A single momentum 4-vector is not enough to describe the energy and momentum in a fluid. We will define a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ symmetric tensor, $T^{\mu\nu}$, called the energy-momentum tensor or stress-energy tensor. It provides all information we need such as pressure, stress, mass and energy density.

The general definition is the following: "it is the flux of four momentum p^μ across a surface of constant x^ν".i.e. T^{00} is the flux of p^0 across a surface of constant x^0, i.e. the energy density in the rest frame of the fluid. and in identical way $T^{0i} = T^{i0}$ is the momentum density in each spatial direction. T^{ij} is the stress.

Off diagonal elements represent shearing terms, like the ones due to viscosity. A diagonal term like T^{11} gives the x-component of the force being exerted per unit of area by a fluid element in the x-direction: this is the x component of the pressure p_x. In general it has three components (no sum here):

$$p^i = T^{ii} \tag{21}$$

2.3 Dust

Non interacting matter (like stars far from each other or galaxies, or a very low pressure gas) can be (one of the quintessential objects loved by physicists) dust. The four-velocity field $U^\mu(x)$ is clearly going to be the constant four-velocity of each particle. If we define the **number-flux-four-vector** to be,

$$N^\mu = nU^\mu, \tag{22}$$

where n is the number density of particles in the rest frame. N^0 is the number of particles measured in any frame (!), while N^i is the flux of particles in the x^i direction. If each particle has the same mass, in the rest frame the energy density of the dust is given by:

$$\rho = mn, \tag{23}$$

Noticing that $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$ we define the energy momentum tensor of dust:

$$T^{\mu\nu}_{dust} = p^\mu N^\nu = mnU^\mu U^\nu = \rho U^\mu U^\nu, \tag{24}$$

In the rest frame of the fluid it is:

$$T^{\mu\nu}_{dust} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

A more general form will be:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}, \tag{26}$$

A non zero pressure perfect fluid, can be written in a more general form, in the rest frame of the fluid:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{27}$$

where a general perfect fluid is defined by a equation of state $p = p(\rho)$. Examples are $p = \rho$, a stiff fluid, $p = 1/3\rho$ a photon gas, or the cosmological constant $p = -\rho$.

2.4 The conservation of energy and momentum in a fluid

There is a nice compact way to express conservation of momentum and energy with the energy momentum tensor.

$$\partial_\mu T^{\mu\nu} = T^{\mu\nu}{}_{,\mu} = 0, \tag{28}$$

For a perfect fluid:

$$\begin{gathered} \partial_\mu T^{\mu\nu} = \partial_\mu(\rho + p)U^\mu U^\nu + \\ (\rho + p)(U^\nu \partial_\mu U^\mu + U^\mu \partial_\mu U^\nu) + \partial^\nu p. \end{gathered} \tag{29}$$

(Note: $\partial^\nu p = \frac{\partial p}{\partial x^\mu} \eta^\mu{}_\nu = p^{,\nu}$).

We can use this expression (which we will set to zero) further if we look at its projections along the fluid (i.e. the direction of the four-velocity) and in a direction orthogonal to the fluid.

We can use the following identity which is a result of the four velocity normalization, $U^\nu U_\nu = -1$,

$$U_\nu \partial_\mu U^\nu = \frac{1}{2}\partial_\mu (U_\nu U^\nu) = 0 \tag{30}$$

Contracting now (21) into U_ν:

$$T^{\mu\nu},_\mu U_\nu = -\rho,_\mu U^\mu - p,_\mu U^\mu - \rho U^\mu,_\mu - pU^\mu,_\mu + p^{,\nu} U_\nu$$

Where I had used (22) and the normalization of the four-velocity. And finally:

$$U_\nu T^{\mu\nu},_\mu = -(\rho U^\nu),_\mu - pU^\mu, \mu. \tag{31}$$

Setting it to 0 in the non-relativistic limit $U^\mu = (1, v^i)$, $\|v^i\| \ll 1$ $p \ll \rho$ (23) becomes

$$(\rho U^0),_0 + (\rho v^i),_i = 0$$

which it is:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{32}$$

Looking at the part orthogonal to the four-velocity, we multiply it by the projection tensor:

$$P^\sigma{}_\nu = \delta^\sigma_\nu + U^\sigma U_\nu. \tag{33}$$

How do we know this is the right projection tensor? Well if $V^\mu = \alpha U^\mu$ then $P^\sigma{}_\nu V^\nu = \alpha \delta^\sigma_\nu U^\nu + \alpha U^\sigma U_\nu U^\nu = 0$ and if W^μ is perpendicular to U^μ it means that $W^\nu U_\nu = 0$, then $P^\sigma{}_\nu W^\nu = \delta^\sigma_\nu W^\nu + U^\sigma U_\nu W^\nu = W^\sigma$ Applying this to (20) we get:

$$P^\sigma{}_\nu \partial_\mu T^{\mu\nu} = (\rho + p) U^\mu \partial_\mu U^\sigma + \partial^\sigma p + U^\sigma U^\mu \partial_\mu p. \tag{34}$$

In the same non-relativistic limit as before if we equal to 0:

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] + \nabla p + \mathbf{v}(\partial_t p + \mathbf{v} \cdot \nabla p) = 0. \tag{35}$$

But if we assume that v is small (non-relativistic limit):

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p. \tag{36}$$

Which is the Euler equation from fluid mechanics that we obtain in (20).

2.5 A review and a more formal recap

Number density in a fluid

We discuss the fact that a fluid is an ensemble of particles, in some sense a continuum, more than a gas but not a rigid body. We saw that dust is the simplest model of a very simple fluid: a collection of particles, all at rest in some Lorentzian frame of reference. The number of particles per unit of volume is one of the physical parameters that clearly reflects relevant information about this collection of particles.
If we measure n as the number density in the rest frame of the fluid, what is the value in another frame moving with velocity in a given direction? We need to consider the volume that this number occupied originally. If it was $\Delta x \Delta y \Delta z$ in the moving frame it will be $\Delta x \Delta y \Delta z \sqrt{1-v^2}$ (the length contracts in the direction of motion -not on the perpendicular ones-). Then the new number density will the frame in motion relative to the rest frame:

$$\frac{n}{\sqrt{1-v^2}} = n' \tag{37}$$

where n' is the number density in the frame in which the particles have velocity v.

Flux in a fluid

How many particles are moving in a given direction in a fluid?
We define *flux* of particles across a surface in a fluid as the number of particles crossing a unit area of that surface in a unit of time. We immediately realize that is also a frame dependent value (depends on the surface orientation and on the frame measured). In the rest frame of the fluid the flux is zero: all particles are at rest. Let's keep in mind that a given volume will be perceived as contracted by an observer in motion, along the direction in which they are moving. To obtain the flux then we need to multiply the number in a given volume by the time they will take to cross a surface S. In a frame moving with velocity v with respect to the comoving frame of the fluid, the number will be obtained as

$$\frac{nv\Delta t A}{\sqrt{1-v^2}} \tag{38}$$

where $v\Delta t A$ is the volume of the fluid. Consequently the flux is

$$flux = \frac{nv}{\sqrt{1-v^2}} \tag{39}$$

The number-flux four vector

Consider the vector $\vec{N}$ defined by

$$\vec{N} = n\vec{U} \tag{40}$$

where $\vec{U}$ is the velocity of the particles. In a frame where the particles have velocity (v^x, v^y, v^z)

$$\vec{U} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v^x}{\sqrt{1-v^2}}, \frac{v^y}{\sqrt{1-v^2}}, \frac{v^z}{\sqrt{1-v^2}} \right). \tag{41}$$

which clearly implies that

$$\vec{N} = \left(\frac{n}{\sqrt{1-v^2}}, \frac{nv^x}{\sqrt{1-v^2}}, \frac{nv^y}{\sqrt{1-v^2}}, \frac{nv^z}{\sqrt{1-v^2}} \right). \tag{42}$$

In Galilean physics the number density is just a scalar, and flux a frame dependent three vector. The number flux four-vector unifies the two concepts in a frame independent four-vector. This is a unification similar to the unification of energy and momentum in the energy-momentum four-vector. If we calculate

$$\vec{N} \cdot \vec{N} = -n^2, \qquad\qquad n = (-\vec{N} \cdot \vec{N})^{1/2} \tag{43}$$

Like rest mass n is a scalar, like energy and inertial mass it is a number that is frame dependent. To avoid ambiguity we always define n to be a scalar number equal to the number density in the Momentarily Comoving Reference Frame (MCRF). And we can also interpret number density as the "timelike" flux.

Flux, frames and one-forms

You should remember Gauss' law of electrodynamics, where the flux of an electric field across a surface is $\vec{E} \cdot \vec{n}$ where here $\vec{n}$ is a unit vector normal to the surface (not related at all to the number density of particles!). In a rather similar manner the flux of particles across a surface of constant ϕ is $\tilde{n} \cdot \vec{N}$ where

$$\tilde{n} = \frac{\tilde{d\phi}}{||\tilde{d\phi}||} \tag{44}$$

and $||\tilde{d\phi}||$ is the magnitude of $\tilde{d\phi}$ which is (for example in a Cartesian coordinate system)

$$\tilde{\phi} = \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \tag{45}$$

and

$$||\tilde{d\phi}|| = |\eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}|^{1/2} \tag{46}$$

If we choose $\phi = t$ we would get $N^0 = n/\sqrt{1-v^2}$, the number density across a surface of constant t.

Note: An inertial frame can be identified by its 4-velocity. But it can also be defined by a 1-form. From the vector $\vec{U}$ we can obtain a one-form in this manner

$$V_\alpha = \eta_{\alpha\beta} V^\beta \tag{47}$$

and in this frame

$$V_0 = -1, \qquad V_i = 0. \tag{48}$$

which is equal to $\tilde{dt}$. This seems to indicate that we could define a frame by giving $\tilde{dt}$.
$\tilde{dt}$ then can be pictured as a set of surfaces of constant t (surfaces of simultaneity). They define a frame up to spacial rotations. Then the energy of particle whose four momentum is $\vec{p}$ is

$$E = < \tilde{dt}, \vec{p} > = p^0 \tag{49}$$

Glossary (from Schutz)

Symbol	**Name**	**Definition**
$\vec{V}$ or $\vec{U}$	four velocity of fluid element	Four-velocity of MCRF
n	Number density	Number of particles per unit of Volume in MCRF
$\vec{N}$	Flux vector	$\vec{n} = n\vec{U}$
ρ	energy density.	Density of **total** mass energy
$\prod$	Internal energy per particle	$\prod = (\rho/n) - m$
ρ_0	rest mass density	$\rho_0 = nm$
T	Temperature	in the MCRF
p	Pressure	in the MCRF
S	Specific Entropy	Entropy per particle.

2.6 A bit of thermodynamics

A perfect fluid is actually defined as a fluid that has no viscosity and no heat conduction in $MCRF$. We can see the effect of the heat conduction if we calculate the energy that could be exchanged in a fluid element either by heat conduction (i.e. absorbing dQ) or by doing work pdV (where V is the usual 3-

$$dE = dQ - pdV \quad or \quad dQ = dE - pdV. \tag{50}$$

If the element contains a total of N particles, and if this number doesn't change we can write:

$$V = \frac{N}{n}, \qquad dV = -\frac{N}{n^2}dn. \tag{51}$$

where n is the number of particles per volume (i.e. density number). But also from the definition of ρ we have $E = \rho V = \rho N/n$ and then from this it results:

$$dE = \rho dV + V d\rho. \tag{52}$$

Combining both equations

$$dQ = \frac{N}{n}d\rho - (\rho + p)\frac{N}{n^2}dn \tag{53}$$

Written in terms of $q = Q/N$

$$ndq = d\rho - \frac{\rho + p}{n}dn \tag{54}$$

So as we do it in thermodynamics, defining $dq = TdS$ we get:

$$d\rho - \frac{\rho + p}{n}dn = nTdS \tag{55}$$

Using

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{56}$$

an calculating the four-divergence $T^{\alpha\beta}{}_{,\beta}$ we have:

$$[(\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta}]_{,\beta} = 0 \tag{57}$$

If we assume particle conservation, i.e. even if the number of particles in a given volume element of the fluid the total number doesn't change, we can write, after defining the flux vector $\vec{N} = n\vec{U}$:

$$\nabla \cdot \vec{N} = N^\alpha{}_{,\alpha} = (nU^\alpha{}_{,\alpha}) = 0 \tag{58}$$

Noticing then that the first term in the energy-momentum conservation law can be written

$$[(\rho + p)U^\alpha U^\beta]_{,\beta} = \left[\frac{\rho + p}{n}U^\alpha U^\beta\right]_{,\beta} \tag{59}$$

$$= nU^\beta \left(\frac{\rho + p}{n}U^\alpha\right)_{,\beta} \tag{60}$$

So, back in (57) we have:

$$nU^{\beta}\left(\frac{\rho+p}{n}U^{\alpha}\right)_{,\beta}+p_{,\beta}\,\eta^{\alpha\beta}=0 \tag{61}$$

Multiplying by U^{α} and summing:

$$U^{\beta}\left[-n\left(\frac{\rho+p}{n}\right)_{,\beta}+p_{,\beta}\right]=0 \tag{62}$$

which is:

$$-U^{\beta}\left[\rho_{,\beta}-\frac{\rho+p}{n}n_{,\beta}\right]=0 \tag{63}$$

But written in the rest frame of the fluid:

$$\frac{d\rho}{d\tau}-\frac{\rho+p}{n}\frac{dn}{d\tau}=0 \tag{64}$$

And comparing with (30)

$$d\rho-\frac{\rho+p}{n}dn=nTdS \tag{65}$$

We see that this is the derivative of the entropy along the lines of flow of the fluid $U^{\alpha}S_{,\alpha}=\frac{dS}{d\tau}$ is:

$$\frac{dS}{d\tau}=0 \tag{66}$$

The flow of a particle-conserving perfect fluid conserves specific entropy $\rightarrow$ **adiabatic**.

One more comment: If we go back to equation (34)

$$P^{\sigma}{}_{\nu}T^{\mu\nu}{}_{,\mu}=(\rho+p)U^{\mu}U^{\sigma}{}_{,\mu}+p^{,\sigma}+U^{\sigma}U^{\mu}p_{,\mu}=0. \tag{67}$$

use that in the $MCRF$ $U^{i}=0$ but $U^{i}{}_{,\beta}\neq 0$ and we get:

$$(\rho+p)U_{i,\beta}U^{\beta}+p_{,i}=0 \tag{68}$$

But now notice that by definition $U_{i,\beta}U^{\beta}$ is the i component of the four acceleration in the direction of $\vec{U}$. So we write:

$$(\rho+p)a_i+p_{,i}=0 \tag{69}$$

or

$$-a_i=\frac{p_{,i}}{\rho+p} \tag{70}$$

For very dense fluids where ρ is comparable to p, the pressure gradient can not overcome the sum of the pressure itself with the density to hold a star in equilibrium.

2.7 Some examples

1

A rod has cross sectional area A and mass per unit length μ. Write down the stress-energy tensor inside the rod when the rod is under tension F (uniform).

$$T^{00} = \mu/A$$

There is no energy flow, so $T^{0i} = 0$.
If z is along the the rod axis, $T^{xy} = T^{xz} = T^{yz} = 0$. So the tension is only in the z direction:

$$\begin{pmatrix} \mu/A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F/A \end{pmatrix}$$

2

The energy -momentum tensor of an electromagnetic field is with $F^{\mu\nu}$ as defined in the first slide:

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \tag{71}$$

a) Show that $T^{00} = (E^2 + B^2)/8\pi$
b) Show that $T^{\alpha}{}_{\alpha} = 0$ i.e. the trace for the energy-momentum tensor of an electromagnetic field is zero.

3

Show that the equation of state for a gas of photons is

$$p = \frac{1}{3}\rho$$

Solution

We know that the trace for an electromagnetic field is zero. In the $MCRF$ it is also

$$T^{\alpha}{}_{\alpha} = -\rho + 3p$$

.

From this

$$p = \frac{1}{3}\rho$$

4

For a perfect fluid with eos $\rho = \rho(n)$ where n is the baryon density, show that $T^{\mu}\mu$ is negative if and only if $d\log\rho/d\log n < 4/3$.

Solution From (30) with $dS = 0$ (iso-entropic fluid),

$$\frac{d\rho}{dt} = \frac{\rho+p}{n}\frac{dn}{dt} \tag{72}$$

Using $\rho = \rho(n)$,

$$\frac{d\rho}{dn}\frac{dn}{dt} = \frac{\rho+p}{n}\frac{dn}{dt} \tag{73}$$

Then dividing by ρ:

$$\frac{d\rho}{\rho} = \frac{\rho+p}{\rho}\frac{dn}{n} \tag{74}$$

so

$$d\log\rho/d\log n = (\rho+p)/\rho \tag{75}$$

Looking at the trace,

$$T^{\mu}{}_{\mu} = (\rho+p)U^{\mu}U_{\mu} + p\eta^{\mu}{}_{\mu} = -(\rho+p) + 4p = 3p - \rho$$

If $T^{\mu}{}_{\mu}$ we can see that $3p-\rho$ is the limit, which we can see by putting $\rho = 3p$ in (49).

5
The velocity of sound in a fluid is $v_s^2 = \partial p/\partial\rho|_{s=cnst}$.
Show that $v_s^2 = \partial\log p/\partial\log n|_{s=ct}p/(\rho+p)$ where $\Gamma_1 = \partial\log p/\partial\log n|_{s=cnst}$ is the adiabatic index.

Solution

We first write:

$$\frac{v_s^2}{\Gamma_1} = \frac{dp/d\rho}{\partial\log p/\partial\log n} = \frac{dp/d\rho}{ndp/pdn}$$

and then,

$$\frac{v_s^2}{\Gamma_1} = \frac{p}{n}\frac{dn}{d\rho}$$

and using (51):

$$v_s^2 = \Gamma_1\frac{p}{\rho+p}$$