Lesson 10
Black Holes

Mario Díaz

The University of Texas Rio Grande Valley

mario.diaz@utrgv.edu

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Hypersurface-orthogonal vector fields

What is the difference between static and stationary? Static means that the solution does not change with time, but furthermore that it is invariant under time reversal. A metric is stationary if there is no dependence on time but there is still "evolution". Let's refine the definition: We will say that a metric is stationary if there is a coordinate system in which the metric is clearly time-independent, i.e.

$$\frac{\partial g_{ab}}{\partial x^0} \stackrel{*}{=} 0, \tag{1}$$

where x^0 is a timelike coordinate. But how do we make this statement coordinate independent? We define a vector field

$$X^a \stackrel{*}{=} \delta^a_0, \tag{2}$$

In this special coordinates, the Lie derivative of the metric is

$$\begin{aligned} \mathcal{L}_{\vec{X}} g_{ab} &= X^c g_{ab,c} + g_{ac} X^c{}_{,b} + g_{bc} X^c{}_{,a} \\ &\overset{*}{=} \delta^c_0 g_{ab,c} = g_{ab,0} = 0 \end{aligned} \tag{3}$$

$\mathcal{L}_{\vec{X}} g_{ab}$ is a tensor, and hence if it vanishes in one system it vanishes in all. X^a is a Killing vector field. And if given a timelike Killing vector field we can always adapt coordinates in which

$$0 = \mathcal{L}_{\vec{X}} g_{ab} \overset{*}{=} g_{ab,0}, \tag{4}$$

and consequently the metric is stationary.
In summary:

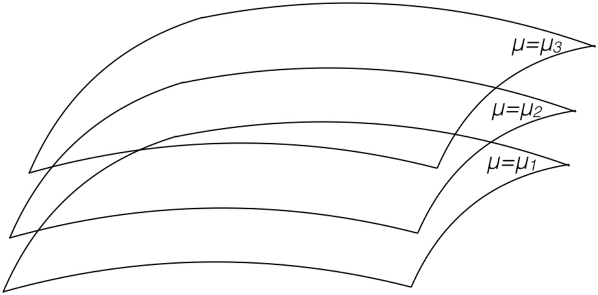
A space-time is said to be **stationary** if and only if it admits a timelike Killing vector field.

Hypersurface-orthogonal vector fields

Let's have a family of hyper surfaces given by:

$$f(x^a) = \mu, \tag{5}$$

μ defines the members of the family.



If we choose two closed points P and Q in one of the hypersurfaces S with coordinates (x^a) and $(x^a + dx^a)$ we get to first order,

$$\mu = f(x^a + dx^a) = f(x^a) + \frac{\partial f}{\partial x^a} dx^a \tag{6}$$

Using (5) we get at P,

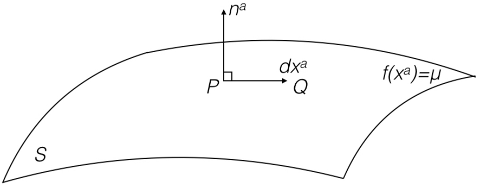
$$\frac{\partial f}{\partial x^a} dx^a = 0 \tag{7}$$

We can define now a covariant vector field n_a

$$n_a \equiv \frac{\partial f}{\partial x^a} \tag{8}$$

Then (7) becomes,

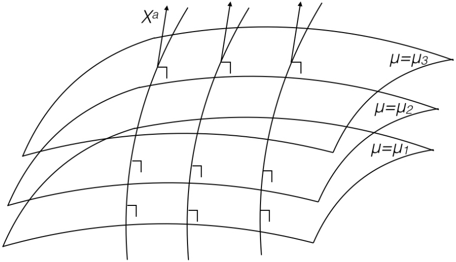
$$n_a dx^a = g_{ab} n^a dx^b = 0 \tag{9}$$



n^a is called the orthogonal vector field to S at P. Any other vector field X^a is called hypersurface-orthogonal if it is everywhere orthogonal to the family of hyper surfaces, i.e. it is proportional to n^a everywhere, i.e.

$$X^a = \lambda(x) n^a \tag{10}$$

Then the orbits of X^a are orthogonal to the family of hypersurfaces.



and X_a is

$$X_a = \lambda f_{,a} \tag{11}$$

and then,

$$X_a \partial_b X_c = \lambda f_{,a} \lambda_{,b} f_{,c} + \lambda^2 f_{,a} f_{,cb} \tag{12}$$

This is equivalent to, taking the totally antisymmetric part of the equation and using covariant derivatives we get,

$$X_{[a} \nabla_b X_{c]} = 0 \tag{13}$$

What it was shown above is that any hypersurface-orthogonal vector field satisfies (13). The converse is (partially) true: any non-null Killing vector satisfying (13) is necessarily hypersurface-orthogonal. We just stated it without proof.

Characterization of coordinates

We need to characterize the Schwarzschild solution in an "invariant" way. Any arbitrary interpretation of coordinates could be meaningless. But there are certain characterizations that can be done: Let's say we have a hypersurface,

$$x^{(a)} = constant \tag{14}$$

A valid question is what is its nature? i.e. timeline, null, or space like at a point. The normal vector field is

$$n_b = \delta_b^{(a)}, \tag{15}$$

it's contravariant version

$$n^c = g^{cb} n_b = g^{cb} \delta_b^{(a)} = g^{c(a)} \tag{16}$$

$$n^2 = n^c n_c = g^{c(a)} \delta_c^{(a)} = g^{(a)(a)}, \quad (no\ sum) \tag{17}$$

If the signature is -2, then (1) at P is **timelike**, **null** or **spacelike** depending on wether $g^{(a)(a)}$ is > 0, $= 0$, or < 0. Of course we will start exploiting the symmetries of the Schwarzschild solution. Thus the metric in coordinates t, r, θ, ϕ) (I'll be using in what follow a signature $(+, -, -, -)$).

$$g^{00} = (1 - 2m/r)^{-1} dt^2, \;\; g^{11} = -(1 - 2m/r) dt^2, g^{22} = -1/r^2, \\ g^{33} = -\frac{1}{r^2 \sin^2 \theta} \tag{18}$$

$x^0 = t$ is timelike and $x^1 = r$ spacelike if $r > 2m$ and $x^2 = \theta$ and $x^3 = \phi$ are spacelike. The Schwarzschild coordinates (t, r, θ, ϕ) are canonical coordinates defined invariantly by the symmetries present.

Points like $\theta = 0, \pi$ are not covered by the Schwarzschild metric because the line element becomes degenerate there.
This is a coordinate problem.
How about $r = 2m$ and $r = 0$? If we look at the scalar invariant

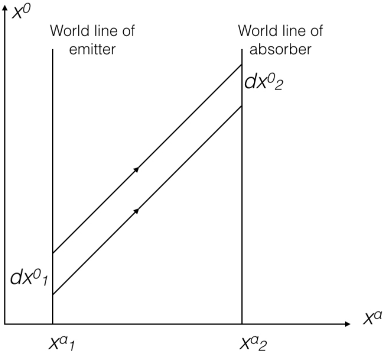
$$R_{abcd}R^{abcd} = 48m^2 r^{-6} \tag{19}$$

We easily see that it is well defined at $r = 2m$. But it does implode at the origin. This singularity is intrinsic and irremovable.
The gravitational redshift is determined by the g_{00} coefficient of the metric (i.e. see d'Inverno) :

$$\bar{\nu}_0 = \nu_0 \left(\frac{g_{00}(x_1^\alpha)}{g_{00}(x_2^\alpha)} \right)^{\frac{1}{2}} \tag{20}$$

in terms of the Schwarzschild metric:

$$\frac{\delta\nu}{\nu} \simeq -\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{21}$$



The picture above illustrates how to calculate eq (21) and shows clearly, that the surface $r = 2m$ defines a sphere of infinite redshift, that cannot communicate with $r > 2m$.

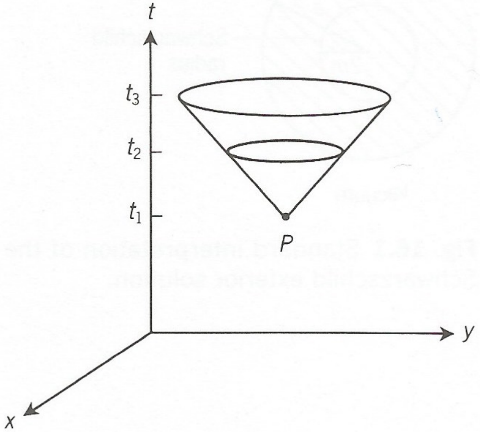
The Schwarzschild solution represents a vacuum solution outside a body of mass m, where outside is defined by $r > r_0 = 2m$ In the case of stars we study interior solutions in Lesson 11. We want here to study the Schwarzschild vacuum solution for all values of r regardless of any source. $r = 2m$ is a null hyper surface dividing the manifold into two disconnected components:

$$I. \quad 2m < r < \infty$$
$$II. \quad 0 < r < 2m.$$

Inside the region II t and r reverse their character with t becoming spacelike and r timelike. The main technique we can use to interpret coordinates is a "spacetime diagram". In particular we want to look at the structure of the local light-cone. This is defined as the geometric figure spanned by the points $x_0^\alpha + dx^\alpha$ in the vicinity of a point x_0^α for which:

$$g_{ab} dx^a dx^b = 0$$

Notice that the "light cone" should be understood in pure "space" as the spherical wave front centered on the point and reaching another point at a later time. if we use only two space dimensions (or one) it is easy to picture it as a cone where the circles at a given time represent this wave fronts.



In Lesson 11 we study the variational method to calculate geodesics, eq (59) and we defined

$$2K \equiv g_{ab}(x)\dot{x}^a\dot{x}^b = \alpha,$$

With

$$ds^2 = \dot{\theta} = \dot{\phi} = 0 \tag{22}$$

With this:

$$2K = (1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 = 0 \tag{23}$$

The dot means derivation respect to a geodesic parameter u. The E-L equation(60) with $a = 0$ is

$$\frac{d}{du}[(1 - 2m/r)\dot{t}] = 0 \tag{24}$$

with integral:

$$(1 - 2m/r)\dot{t} = k \tag{25}$$

So in (23) $\dot{r}^2 = k^2$ and then $\dot{r} = \pm k$ We can try to find the solution in the form $t = t(r)$,

$$\frac{dt}{dr} = \frac{dt/du}{dr/du} = \frac{\dot{t}}{\dot{r}}, \tag{26}$$

From (25) and from $\dot{r} = \pm k$,

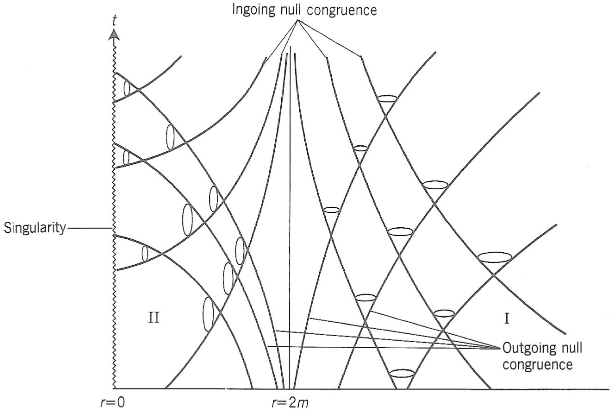
$$\frac{dt}{dr} = \frac{r}{r-2m}, \tag{27}$$

And integrating,

$$t = r + 2m \ln|r-2m| + constant \tag{28}$$

In region I from (27) $r > 2m \Rightarrow \frac{dr}{dt} > 0$, and r increases as t increases. Then curves (28) define a congruence of **outgoing** radial null geodesics. The opposite case gives the congruence of **ingoing** radial null geodesics.

$$t = -(r + 2m \ln|r-2m| + constant). \tag{29}$$



Plot from d'Inverno's book

This plot shows θ and ϕ suppressed but due to symmetry the plot would look the same for any value of them. The local light cones would flip over when crossing regions due to the different nature of the coordinates. An observer inside the BH is forced to fall towards $r = 0$.

Infalling particles

A particle moving radially into the B-H will move on a timelike geodesic

$$(1 - 2m/r)\dot{t} = k \quad (30)$$

$$(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 = 1 \quad (31)$$

The dot means derivative respect to the proper time. If we choose initial condition corresponding to dropping the particle from ∞ with $v_0 = 0$, meaning at large r $\dot{t} \simeq 1$ we'll get

$$\left(\frac{d\tau}{dr}^2\right) = \frac{r}{2m} \quad (32)$$

Taking the negative sq rt and integrating, we find where the particle is at r_0 at time τ_0:

$$\tau - \tau_0 = \frac{2}{3(2m)^{\frac{1}{2}}}(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}) \quad (33)$$

If we describe motion in terms of t

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\frac{r}{2m}^{\frac{1}{2}} \left(1\frac{2m}{r}\right)^{-1} \tag{34}$$

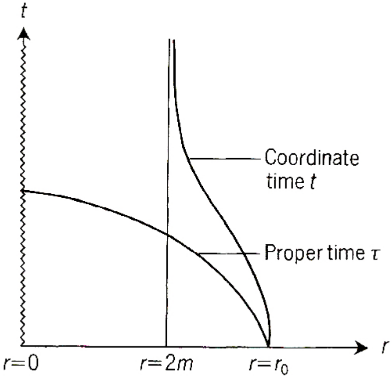
Integrating, we obtain

$$\begin{aligned} t - t_0 = & -\frac{2}{3(2m)^{\frac{1}{2}}}\left(r^{\frac{3}{2}} - r_0^{\frac{3}{2}} + 6mr^{\frac{1}{2}} - 6mr_0^{\frac{1}{2}}\right) \\ & + 2m \ln \frac{[r^{\frac{1}{2}} + (2m)^{\frac{1}{2}}][r_0^{\frac{1}{2}} - (2m)^{\frac{1}{2}}]}{[r_0^{\frac{1}{2}} + (2m)^{\frac{1}{2}}][r^{\frac{1}{2}} - (2m)^{\frac{1}{2}}]} \end{aligned} \tag{35}$$

Whenever r_0 and r are much larger than $2m$ (33) and (35) give pretty much the same result. But if r is close to $2m$ we get,

$$r - 2m = (r_0 - 2m)e^{-(t-t_0)/2m} \tag{36}$$

and we can easily infer that as $t \to \infty \Rightarrow r - 2m \to 0$



Radially infalling particle in coordinates t and τ
Plot from d'Inverno's book

Eddington-Finkelstein coordinates

Following (28) we change coordinates, and for $r > 2m$,

$$t \to \bar{t} = t + 2m \ln(r - 2m) \tag{37}$$

In the new $(\bar{t}, r, \theta, \phi)$ (28) becomes:

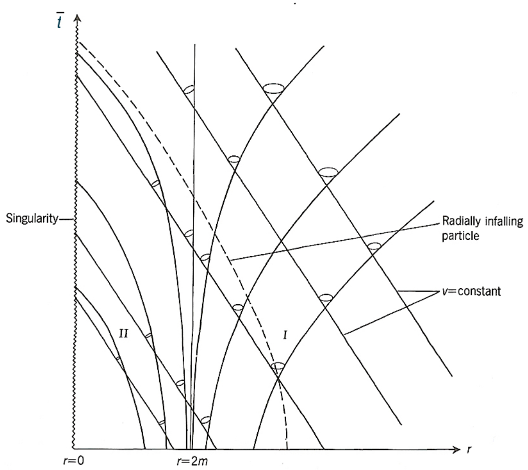
$$\bar{t} = -r + constant, \tag{38}$$

This is a straight line at -45° from the r axis. From it,

$$d\bar{t} = dt + \frac{2m}{r - 2m} dr, \tag{39}$$

and the new metric,

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\bar{t}^2 - \frac{4m}{r} d\bar{t} dr - (1 + 2mr)\, dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{40}$$



Schwarzschild solution in Eddington-Finkelstein coordinates
Plot from d'Inverno's book

A few comments: E-F metric is regular everywhere but $r = 0$, but it is no longer time-symmetric. Outgoing radial null geodesics can be described as straight lines with

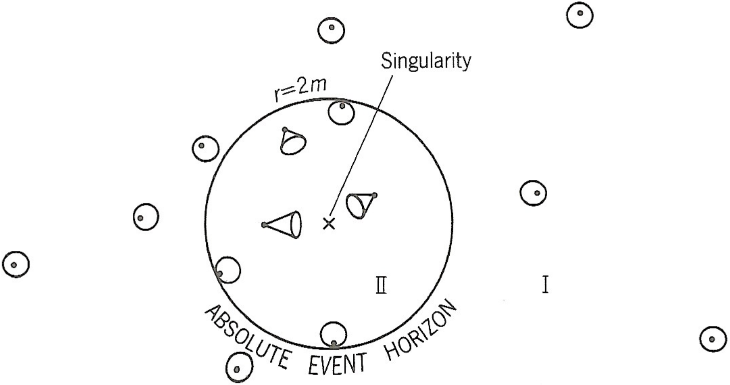
$$t \to t^* = t - 2m \ln(r - 2m), \tag{41}$$

(40) using a null coordinate (advanced parameter) becomes

$$v = \bar{t} + r, \tag{42}$$

$$ds^2 = (1 - 2m/r)dv^2 - dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{43}$$

Event horizon

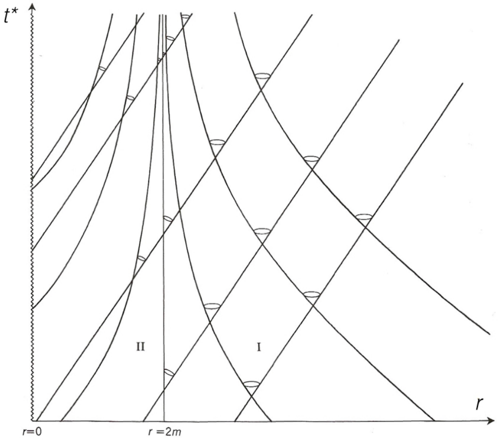


Equatorial plane cut from the previous figure, fixed $\bar{t}$ in Eddington-Finkelstein coordinates
Drawing from d'Inverno's book

It is clear now that $r = 2m$ acts as a one way membrane: only future directed timelike and null curves cross from outside. No future directed timelike or null curves can escape from the outside. This is an absolute event horizon. We can use $w - t * - r$ and (40) is " time-reversed",

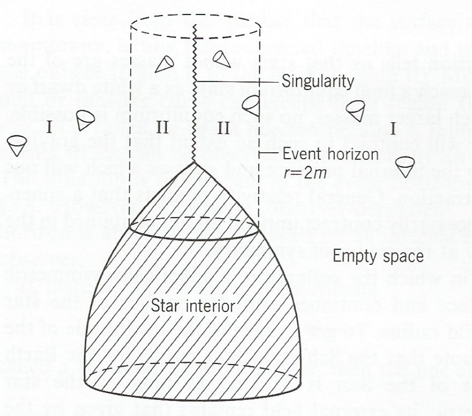
$$ds^2 = (1 - 2m/r)dw^2 + 2dwdr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (44)$$

This is once more regular for $0 < r < \infty$.
The membrane acts in the opposite time direction: only past-directed timelike or null curves cross from the outside to the inside.



S-T Diagram in time reversed Eddington-Finkelstein coordinates
Plot from d'Inverno's book

The following drawing shows a gravitational collapse obtained in $\bar{t}$ and r coordinates made by rotating r around the $\bar{t}$ axis, i.e. one dimension suppressed.



Gravitational collapse
Drawing from d'Inverno's book

Einstein Rosen bridge and wormholes

Let's consider now the Schwarzschild solution in null coordinates advanced and retarded coordinates,

$$v = t + r + 2m\ln(r/2m - 1), u = t - r - 2m\ln(r/2m - 1) \quad (45)$$

Then,

$$ds^2 = -(1 - 2m/r)dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (46)$$

If we introduce coordinates

$$z = 1/2(e^{v/4m} + e^{-u/4m}), \quad w = 1/2(e^{v/4m} - e^{-u/4m}) \quad (47)$$

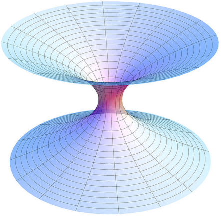
$$ds^2 = \frac{32m^3}{r}e^{-r/2m}(dz^2 - dw^2) + r^2(z,w)d\Omega^2 \quad (48)$$

which is the Schwarzschild metric in Kruskal-Szekeres coordinates.

If we now consider a S-T with coordinates z, w, θ and ϕ we can look at the space like slice $w = 0$ and put $\theta = \pi/2$, then

$$ds^2 = \frac{32m^3}{r} e^{-r/2m} dz^2 + r^2 d\phi^2 = \left(\frac{r}{r-2m}\right) dr^2 + r^2 d\phi^2 \quad (49)$$

This is the metric on a surface which is a paraboloid of revolution (i.e. by rotating the parabola $y = \frac{z^2}{8m} + 2m$ around the z axis).



Wormhole

If we want to find a solution with spherical symmetry which corresponds to a mass m and electric charge q we need to solve the Einstein's eqs for an energy momentum tensor that would not be zero. The energy momentum tensor of an electromagnetic field is traceless, so Einstein's eqs become:

$$R_{ab} = 8\pi T_{ab} \tag{50}$$

$$T_{ab} = \frac{1}{4\pi}(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}) \tag{51}$$

$$\nabla_b F^{ab} = 0 \tag{52}$$

$$\partial_{[a} F_{bc]} = 0. \tag{53}$$

$$F_{ab} = E(r) \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{54}$$

With a metric,

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (55)$$

where ν and λ are functions of r only. Then solving for the metric (54) in (50) with the constraints (51), (52) and (53) we get just one equation:

$$\frac{d}{dr}(e^{-\frac{1}{2}(\nu+\lambda)} r^2 E) = 0, \quad (56)$$

Integrating we get,

$$E = e^{-\frac{1}{2}(\nu+\lambda)} q/r^2, \quad (57)$$

where q is a constant of integration. Our solution should be asymptotically flat for large r, i.e.

$$\nu, \lambda \to 0 \quad \text{as} \quad r \to \infty \quad (58)$$

And then $E \sim q/r^2$ asymptotically which justifies the choice of q. Using (54) and (56) we find in (51) that the 00 and 11 eqs,

$$\frac{d}{dr}(\nu + \lambda) = 0 \tag{59}$$

which from (58) implies $\lambda = -\nu$. Using the 22 eq.

$$\frac{d}{dr}(re^{\nu}) = 1 - q^2/r^2, \tag{60}$$

which gives

$$e^{\nu} = 1 - 2m/r + q^2/r^2, \tag{61}$$

with m a constant of integration (like q), and then the **Reissner-Nordstrøm** solution:

$$ds^2 = \left(1 - 2m/r + q^2/r^2\right) dt^2 - \left(1 - 2m/r + q^2/r^2\right)^{-1} dr^2 \\ -r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{62}$$

Singularities in Reissner-Nordstrøm

Looking at the polynomial

$$Q = r^2 + q^2 - 2mr \tag{63}$$

it has discriminant $m^2 - q^2$ and hence three different possibilities.Let $r_\pm$ be,

$$r_\pm = m \pm (m^2 - q^2)^{\frac{1}{2}} \tag{64}$$

The line element is regular in these regions,

$$\begin{aligned} I.r_+ < r < \infty, \\ II.r_- < r < r_+, \\ III.0 < r < r_-. \end{aligned}$$

IF $Q^2 = M^2$ only regions I and III exist. The regions are separated by the null hypersurfaces $r = r_+$ (like in Schwarzschild $r = 2m$) and $r = r_-$

Restricting to $q^2 < m^2$ for $r > r_+$ we define,

$$\bar{t} = t + \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2}{r_+ - r_-} \ln(r - r_-) \tag{65}$$

The metric then becomes, after defining f

$$f = 2m/r - q^2/r^2 \tag{66}$$

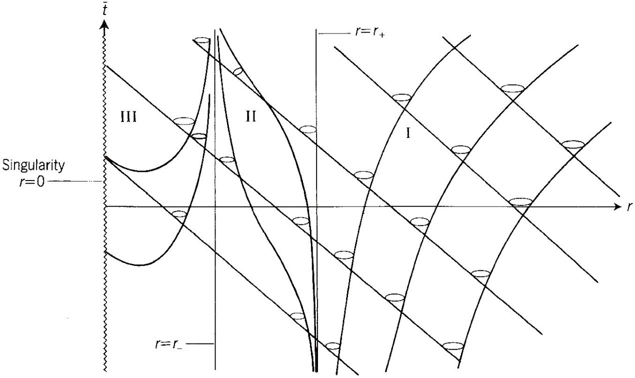
$$ds^2 = (1-f)d\bar{t}^2 - 2fd\bar{t}dr - (1+f)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{67}$$

The ingoing null geodesics are

$$\bar{t} + r = constant \tag{68}$$

and the outgoing ones

$$\frac{d\bar{t}}{dr} = \frac{1+f}{1-f}, \tag{69}$$



Reissner-Nordstrøm $q^2 < m^2$ solution in advanced Eddington-Finkelstein coordinates
Plot from d'Inverno's book

Notice that in region *III* neutral particles cannot reach the singularity.

A Penrose diagram is a two-dimensional diagram that describes the causal relations between different points in spacetime. It is is similar to a Minkowski diagram where the vertical axis represents time, the horizontal axis space, and lines at 45° light rays. The basic idea is to "squeeze" the entire space-time in a finite region by performing a conformal transformation.

Minkowski

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{70}$$

We define

$$u = t - r, \qquad v = t + r \tag{71}$$

$$ds^2 = -dudv + \frac{1}{4}(u - v)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{72}$$

We introduce a conformal factor,

$$\Omega^2 = \frac{1}{(1+u^2)(1+v^2)} \tag{73}$$

and this gives

$$d\bar{s}^2 = \Omega^2 ds^2 = -\frac{dudv}{(1+u^2)(1+v^2)} + \frac{1}{4}\frac{(u-v)^2}{(1+u^2)(1+v^2)}(d\theta^2 + \sin^2\theta d\phi^2) \tag{74}$$

And now define $u = \tan p$ and $v = \tan q$, so that,

$$\frac{(u-v)^2}{(1+u^2)(1+v^2)} = \sin^2(p-q) \tag{75}$$

And then (69) becomes

$$d\bar{s}^2 = -dpdq + \frac{1}{4}\sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (76)$$

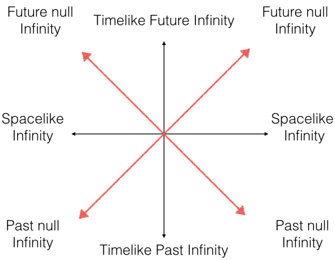
with

$$-\pi \leqslant p, \quad q \leqslant \pi \qquad (77)$$

We can describe the geometric properties of this representation better defining

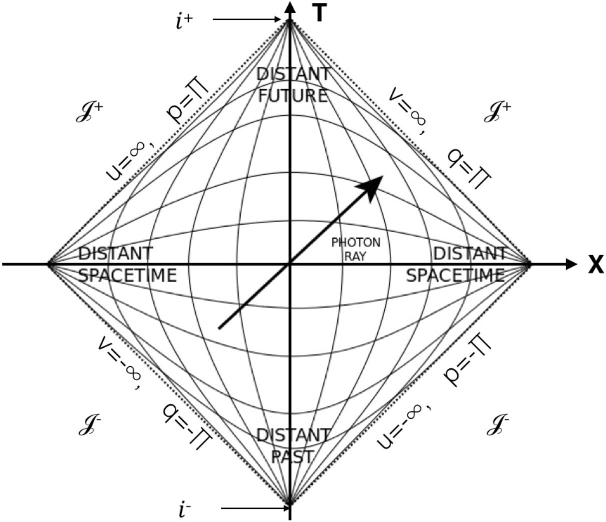
$$p = T - R, \qquad q = T + R \qquad (78)$$

with $\mathscr{I}^+$ being the null future infinity and
$\mathscr{I}^-$ the null past infinity. Timelike future infinity is $\imath^+$ and the timelike past infinity is $\imath^-$,
$\imath^0$ represents space like infinity.



In the Penrose diagram,

$$
\begin{aligned}
\imath^{+} &: T = \pi, \\
\imath^{-} &: T = -\pi, \\
\imath^{0} &: q = -p = \pi, \quad R = \pi; \quad p = -q = \pi, \quad R = -\pi \qquad (79) \\
\mathscr{I}^{+} &: T \pm R = \pi, \\
\mathscr{I}^{-} &: T \pm R = -\pi,
\end{aligned}
$$



Conformal Minkowski

Let's consider now,again, the Schwarzschild solution in null coordinates advanced and retarded coordinates,

$$v = t + r + 2m \ln\left(\frac{r}{2m} - 1\right) \quad u = t - r - 2m \ln\left(\frac{r}{2m} - 1\right) \tag{80}$$

$$ds^2 = -(1 - 2m/r)dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{81}$$

We introduce now new U, V

$$U = -4me^{-u/4m}, \qquad V = 4me^{v/4m} \tag{82}$$

$$\left(1 - \frac{r}{2m}\right) dudv = \frac{2m}{r} e^{-r/2m} dUdV \tag{83}$$

So now (81) becomes,

$$ds^2 = -\frac{2m}{r} e^{-r/2m} dUdV + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{84}$$

The range of definition now changes

$$u = -\infty, \quad U = -\infty : v = -\infty, V = 0$$
$$u = 0, \quad U = -4m : v = 0, V = 4m$$
$$u = \infty, \quad U = 0 : v = \infty, V = \infty$$

and we also have

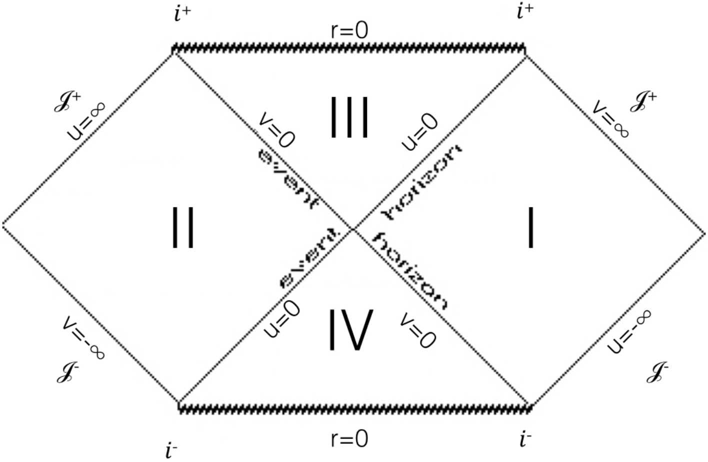
$$-\infty \leqslant U \leqslant 0, \quad 0 \leqslant V \leqslant \infty$$

But if we add to the diagram we would get from using these coordinates and range of values another diagram

$$U \geqslant 0, \quad 0 \leqslant V \leqslant 0$$

we would get a diagram that represents Schwarzschild solution over the whole range in the Kruskal coordinates,

$$-\infty \leqslant U, V \leqslant \infty$$



Conformal Schwarzschild in Kruskal coordinates

Null tetrads

We will use four LI vector fields e_i^a, where i labels the vectors. We define the **frame metric**,

$$g_{ij} = g_{ab} {e_i}^a {e_j}^b \tag{85}$$

We can define the inverse

$$g_{ij} g^{jk} = \delta_i^k \tag{86}$$

And then,

$$g_{ab} = g_{ij} {e^i}_a {e^j}_b, \tag{87}$$

Let's define four unit vectors, v^a timelike and three spacelike i^a, j^a, and k^a, and the frame metric (85) is the Minkowski metric. We define now:

$$e_0{}^a = l^a = \frac{1}{\sqrt{2}}(v^a + i^a), \tag{88}$$

$$e_1{}^a = n^a = \frac{1}{\sqrt{2}}(v^a - i^a) \tag{89}$$

Of course $l^a l_a = n^a n_a = 0$ and also satisfy $l^a n_a = 1$. We finally take $e_2{}^a = j^a$ and $e_3{}^a = k^a$ and we get the following frame metric,

$$g_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{90}$$

and the following complex null vector and its conjugate

$$m^a = \frac{1}{\sqrt{2}}(j^a + ik^a), \quad \bar{m}^a = \frac{1}{\sqrt{2}}(j^a - ik^a) \tag{91}$$

Of course, $m^a m_a = \bar{m}^a \bar{m}_a = 0$, and $m^a \bar{m}_a = -1$. and now we choose the null tetrad,

$$({e_0}^a, {e_1}^a, {e_2}^a, {e_3}^a) = (l^a, n^a, m^a, \bar{m}^a), \tag{92}$$

which has the associated frame metric,

$$g_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{93}$$

And then (87) can be written,

$$g_{ab} = l_a n_b + l_b n_a - m_a \bar{m}_b - m_b \bar{m}_a, \tag{94}$$

and,

$$g^{ab} = l^a n^b + l^b n^a - m^a \bar{m}^b - m^b \bar{m}^a, \tag{95}$$

The non-zero components of the Schwarzschild solution in Eddington-Finkelstein coordinates are

$$g^{01} = -1, g^{11} = -\left(1 - \frac{2m}{r}\right), g^{22} = -\frac{1}{r^2}, g^{33} = -\frac{1}{r^2 \sin^2 \theta}. \quad (96)$$

which can be written in terms of the following tetrad,

$$\begin{aligned} l^a &= (0, 1, 0, 0) = \delta_1^a, \\ n^a &= (-1, -\frac{1}{2}(1 - 2m/r), 0, 0) = -\delta_0^a - \frac{1}{2}(1 - 2m/r)\delta_1^a, \\ m^a &= \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right) = \frac{1}{\sqrt{2}r}\left(\delta_2^a + \frac{i}{\sin\theta}\delta_3^a\right) \end{aligned} \quad (97)$$

and we do the following complex transformation on the tetrad

$$v \to v' = v + ia\cos\theta, r \to r' = r + ia\cos\theta, \theta \to \theta', \phi \to \phi' \quad (98)$$

Making the requirement that v' and r' are real,

$$
\begin{aligned}
l'^a &= \delta_1^a, \\
n'^a &= -\delta_0^a - \frac{1}{2}\left(1 - \frac{2mr'}{r'^2 + a^2\cos^2\theta}\right)\delta_1^a, \\
m'^a &= \frac{1}{\sqrt{2(r' + ia\cos\theta)}}\left(-ia\sin\theta(\delta_0^a + \delta_1^a) + \delta_2^a + \frac{i}{\sin\theta}\delta_3^a\right).
\end{aligned}
\tag{99}
$$

This is the Kerr solution and the contravariant form can be obtained using (95).

Let's define

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{100}$$

Then

$$ds^2 = \left(1 - \frac{2mr}{\rho^2}\right) dv^2 - 2dvdr + \frac{2mr}{\rho^2}(2a\sin^2\theta)dvd\bar{\phi} + 2a\sin^2\theta drd\bar{\phi}$$
$$-\rho^2 d\theta^2 - \left((r^2+a^2)\sin^2\theta + \frac{2mr}{\rho^2}(a^2\sin^4\theta)\right) d\bar{\phi}^2, \tag{101}$$

where $(v, r, \theta, \bar{\phi})$ are related to (t, r, θ, ϕ) by,

$$dv = d\bar{t} + dr = dt + \frac{2mr + \Delta}{\Delta} dr \tag{102}$$

$$d\bar{\phi} = d\phi + \frac{a}{\Delta} dr \tag{103}$$

$$\Delta = r^2 - 2mr + a^2 \tag{104}$$

With this we can obtain Kerr's solution in Boyer-Lindquist coordinates,

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - adt]^2 - \\ -\frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \quad (105)$$

and the following is the original metric in Kerr form with $(\bar{t}, x, y, z)$

$$ds^2 = d\bar{t} - dx^2 - dy^2 - dz^2 - \\ -\frac{2mr^3}{r^4+a^2z^2}\left(d\bar{t} + \frac{r}{a^2+r^2}(xdx+ydy) + \right. \\ \left. + \frac{a}{a^2+r^2}(ydx - xdy) + \frac{z}{r}dz\right)^2, \quad (106)$$

$$\left.\begin{aligned} \bar{t} &= v - r, \\ x &= r\sin\theta\cos\phi + a\sin\theta\sin\phi, \\ y &= r\sin\theta\sin\phi - a\sin\theta\cos\phi, \\ z &= r\cos\theta \end{aligned}\right\} \tag{107}$$

The metric has the form,

$$ds^2 = \eta_{ab}dx^a dx^b - \lambda l_a l_b dx^a dx^b, \tag{108}$$

$$\eta_{ab}l^a l^b = 0, \quad \lambda = \frac{2mr^3}{r^4 + a^2 z^2} \tag{109}$$

$$l_a = \left(1, \frac{rx + ay}{a^2 + y^2}, \frac{ry - ax}{a^2 + y^2}, \frac{z}{r}\right) \tag{110}$$

In Schwarzschild,

$$\lambda = 2m/r, \qquad l_a = (1, x/r, y/r, z/r). \tag{111}$$

Basic properties

Starting with the Boyer-Lindquist:

- If we set $a = 0$ we regain the Schwarzschild solution. So m is the geometrical mass.
- Metric coefficients are independent of t and ϕ so the solution has $\partial/\partial t$ and $\partial/\partial\phi$ as Killing vector fields. i.e. it is stationary and axially symmetric.
- The solution is invariant under simultaneous $t \to -t$ and $\phi \to -\phi$. Similarly it is also under $t \to -t$ and $arightarrow - a$ which indicates that it represents a spinning object with spin a.
- The standard polar coordinate is $R^2 = x^2 + y^2 + z^2 = r^2 + a^2 \sin^2\theta$. For $r \gg a$ $R = r + \frac{a^2 \sin^2\theta}{2r} + \cdots$ which shows that as $r \to \infty$ also $R \to \infty$ and then $g_{ab} \to \eta_{ab}$ which shows that Kerr is asymptotically flat.

- All efforts to interpret exactly the meaning of a, although not all having the same mathematical and physical rigor, conclude nonetheless that the parameter a in the Kerr solution is related to the angular velocity and that ma is related to the angular momentum.
- Expanding (106) in powers of $1/R$ we get:

$$ds^2 = \left(1 - \frac{2m}{R} + \cdots\right) dt^2 - \frac{4ma}{R^3}(xdy - ydz)dt + \cdots \tag{112}$$

- Which is a strong suggestion that ma corresponds to the angular momentum of the hole.

- Calculating the Riemann invariant $R^{abcd}{}_{abcd}$ we can see that the only essential singularity occurs at $\rho = 0$.
- $\rho = 0$ implies

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0 \tag{113}$$

- This implies that the singularity is located where both $r = 0$ and $\cos\theta = 0$.
- This occurs when $z = 0$ and $x^2 + y^2 = a^2$.
- The singularity then is a ring of radius a located in the plane $z = 0$.

- The surfaces of infinite redshift occur when

$$g_{00} = (r^2 - 2mr + a^2 \cos^2\theta)/\rho^2, \tag{114}$$

- The surfaces are then

$$r = r_{s_\pm} = m \pm (m^2 - a^2 \cos^2\theta)^{1/2} \tag{115}$$

- the Schwarzschild metric is neatly obtained in the limit $a \to 0$.
- we find in general two surfaces of infinite redshift.
- To study horizons it is convenient to look at the Killing vector field $X^a = (1, 0, 0, 0)$ with magnitude $X^2 = g_{00}$.
- This vector is then timelike outside S_+ and inside S_-, null on both S_+ and S_-.
- We look then for the event horizon by looking for the hypersurfaces where $r = const$ becomes null, i.e. where $g^{11} = 0$.

- Using Boyer-Lindquist coordinates (105)

$$g^{11} = -\frac{\Delta}{\rho^2} = -\frac{r^2 - 2mr + a^2}{r^2 + a^2 \cos^2\theta} \tag{116}$$

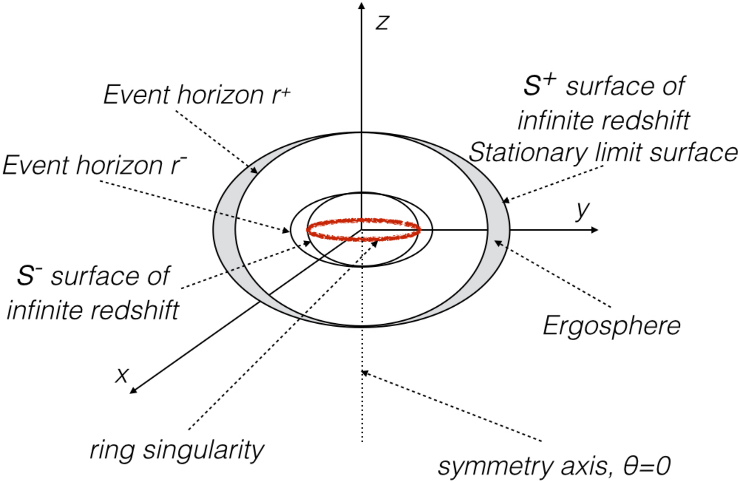
- which implies that $g^{11} = 0$ when $r^2 - 2mr + a^2 = 0$
- if we assume that $a^2 < m^2$ this results in two null event horizons.

$$r = r_{\pm} = m \pm (m^2 - a^2)^{1/2}. \tag{117}$$

- Kerr is regular in three regions:

$$\begin{aligned} I. \qquad & r_+ < r < \infty \\ II. \qquad & r_- < r < r_+, \\ III. \qquad & 0 < r < r_- \end{aligned} \tag{118}$$

- when $a \to 0$ the two event horizons reduce to $r = 2m$ and $r = 0$, i.e. in Schwarzschild the surfaces of infinite redshift and the event horizons coincide.
- in Kerr $r = r_+$ is completely within the sphere S_+. The region between the two spheres (notice that the distance to $r = 0$ of S_+ is modulated by a factor $a \cos^2 \theta$) is called the **ergosphere**.
- Kerr is no longer spherically symmetric so we no longer expect it ti have radially null geodesics.
- the next slide show a diagram of the different event horizons and the surfaces of infinite redshift in Kerr.



The event horizons, the essential singularity and the surfaces of infinite redshift in Kerr