Lesson 13

Cosmology

A phenomenological approach

Mario Diaz

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The Friedmann equation

We assume $\vec{u} = (1, 0, 0, 0)$ and work with

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{1}$$

Writing it with one index up

$$T^\mu{}_\nu = diag(-\rho, p, p, p), \tag{2}$$

and the trace

$$T = T^\mu{}_\mu = -\rho + 3p. \tag{3}$$

and the 0 component of the conservation of energy equation

$$
\begin{aligned}
0 =& \nabla_\mu T^\mu_0 \\
=& \partial T^\mu_0 + \Gamma^\mu{}_{\mu\lambda} T^\lambda{}_0 + \Gamma^\lambda{}_{\mu 0} T^\mu{}_\lambda \\
=& -\partial_0 p - 3\frac{\dot{a}}{a}(\rho + p)
\end{aligned} \tag{4}
$$

where the metric is

$$
ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{5}
$$

Choosing an equation of state

$$
p = w\rho \tag{6}
$$

where w is a constant.

With this (4) becomes

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + w) \tag{7}$$

Integration gives

$$\rho \propto a^{-3(1+w)} \tag{8}$$

Energy conditions: If we take a completely arbitrary metric, we can construct the Einstein tensor. Then, we make the energy momentum tensor equal to it, and voila.., our metric satisfies the Einstein equations!

The Bianchi identities will be automatically satisfied and it will be automatically conserved. If we use particular sources we can restrict the energy momentum tensor. But it has relevance to question what conditions should an energy momentum tensor satisfy.

- **The weak energy condition**
  States that $T_{\mu\nu}t^{\mu}t^{\nu} \geqslant 0$ for all timelike vectors t^{μ}, which implies $\rho \geqslant 0$ and $\rho + p \geqslant 0$.

- **The null energy condition**
  $T_{\mu\nu}l^{\mu}l^{\nu} \geqslant 0$ for all null vectors l^{μ}. ρ may be negative provided that $\rho + p \geqslant 0$.

- **The dominant energy condition**
  $T_{\mu\nu}t^{\mu}t^{\nu} \geqslant 0$ for all timelike vectors t^{μ} and $T_{\mu\nu}t_{\mu}$ is a non-spacelike vector.
  Equivalent to $T_{\mu\nu}T^{\nu}{}_{\lambda}t^{\mu}t^{\lambda} \leqslant 0$. $\rightarrow \rho \geqslant |p|$

- **The null dominant energy condition**
  $T_{\mu\nu}l^{\mu}l^{\nu} \geqslant 0$ for all null vectors l^{μ} and $T_{\mu\nu}l_{\mu}$ is a non-space vector. This allows for $p = -\rho$.

- **The strong energy condition**
  States that $T_{\mu\nu}t^\mu t^\nu \geqslant \frac{1}{2}T^\lambda{}_\lambda t^\sigma t_\sigma$ for all timelike vectors t^μ. $\rightarrow \rho + p \geqslant 0$ and $\rho + 3p \geqslant 0$. Gravitation is attractive.

Back to (8) what are the possible values of w? NDEC which allows $p = -\rho$ implies $|w| \leqslant 1$.

The two best examples are matter and radiation. A matter dominated universe obeys

$$\rho_M \propto a^{-3} \tag{9}$$

For a radiation dominated universe we have that the equation of state gives

$$p_R = \frac{1}{3}\rho_R \tag{10}$$

$$\rho_R \propto a^{-4} \tag{11}$$

a result of the individual photons loosing also energy as a^{-1} as they redshift. We believe that today the radiation energy density is less than that of matter with

$$\frac{\rho_M}{\rho_R} \sim 10^3 \tag{12}$$

In the past the universe was much smaller and ρ_R would have dominated at early times.

Vacuum energy can be modeled as a perfect fluid with EoS

$$p_\Lambda = -\rho_\Lambda \tag{13}$$

The energy density is constant and $\rho_\Lambda \propto a^0$. Since both ρ_M and ρ_R decrease with a if the universe is expanding, eventually the universe will become vacuum dominated. The deSitter and anti-de Sitter are vacuum dominated solutions.

Einstein's equations can be cast as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{14}$$

(throughout this part we keep G and use $c = 1$). The $\mu\nu = 00$ equation is

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) \tag{15}$$

while the $\mu\nu = ij$ eqs are

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \tag{16}$$

Using (15) to eliminate the second derivatives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{17}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{18}$$

(17) and (18) are the Friedmann equations we already saw. as we already discussed

$$H = \frac{\dot{a}}{a} \tag{19}$$

is the Hubble parameter. Now is called H_0. Current measurements give $H_0 = 70 \pm 10 km/sec/Mpc$.

It is common to parametrize

$$H_0 = 100\ h\ km/sec/Mpc \tag{20}$$

where $h \approx 0.7$. The Hubble length

$$\begin{aligned} d_H =& {H_0}^{-1} c = 9.25 \times 10^{27} cm \\ =& 3.08 \times 10^3 h^{-1} Mpc, \end{aligned} \tag{21}$$

the Hubble time

$$t_H = {H_0}^{-1} = 3.09 \times 10^{17} h^{-1} sec = 9.78 \times 10^9 h^{-1} yr. \tag{22}$$

The deceleration parameter (rate of the expansion):

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{23}$$

A very important quantity is the density parameter

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}} \tag{24}$$

where

$$\rho_{crit} = \frac{3H^2}{8\pi G} \tag{25}$$

Remember that (18) can be written

$$\Omega - 1 = \frac{k}{H^2 a^2}. \tag{26}$$

Then the sign of k depends on the value of ρ

$$\begin{array}{llll} \rho < \rho_{crit} & \Omega < 1 & k < 0 & \text{open} \\ \rho = \rho_{crit} & \Omega = 1 & k = 0 & \text{flat} \\ \rho > \rho_{crit} & \Omega > 1 & k > 0 & \text{closed.} \end{array}$$

The density tells then which of the FRW geometries describe the universe.

Evolution of the Universe

One way of proceeding is modeling each component of ρ

$$\rho_i = \rho_{i0} a^{-n_i}, \tag{27}$$

This is equivalent to assume

$$w_i = \frac{1}{3} n_i - 1, \tag{28}$$

We can also think that the spatial curvature contributes with a fictitious density

$$\rho_S \equiv -\frac{3k}{8\pi G a^2}, \tag{29}$$

with a "density parameter"

$$\Omega_S = -\frac{k}{H^2 a^2}, \tag{30}$$

Of course this is not an energy density: it's just helpful notation. The relevant sources we can consider

	w_i	n_i
matter	0	3
radiation	1/3	4
curvature	-1/3	2
vacuum	-1	0

In these variables (18) can be written

$$H^2 = \frac{8\pi G}{3} \sum_{i(all)} \rho_i \tag{31}$$

where we sum over all ρ_i and ρ_S as well. Dividing by H^2 both sides,

$$1 = \sum_{i(all)} \Omega_i \tag{32}$$

If we separate Ω_S we get

$$\Omega_S = 1 - \Omega, \tag{33}$$

If all ρ_i's are nonnegative, because of (31) the universe will never undergo a transition from expanding to contracting if $\sum_{i(all)} \rho_i \neq 0$. Taking the time derivative of H

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2, \qquad (34)$$

plugging it in the two Friedmann eqs (17) and (18)

$$\dot{H} = -4\pi G \sum_{i(all)} (1 + w_i)\rho_i \qquad (35)$$

Since we assumed $|w_i| \leqslant 1$ if all the ρ_i's are non-negative we will always have $\dot{H} \leqslant 0$. Which implies that the universe keeps expanding but the expansion rate decreases with time. While ρ_i for matter and radiation are non-negative we could have ρ_S

and ρ_Λ which could be negative and can change the sign of the Hubble parameter. An example is the de Sitter metric

$$ds^2 = dt^2 + \alpha^2 \cosh^2(t/\alpha)[d\chi^2 + \sinh^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2] \tag{36}$$

where

$$-u^2 + x^2 + y^2 + z^2 + w^2 = \alpha^2 \tag{37}$$

is a hyperboloid embedded in the 5-D Minkowski metric. It describes a universe with a positive vacuum energy but also positive spatial curvature. It initially collapses to a turning point and then begins to expand thereafter.

If we consider solutions with $\rho \propto a^{-n}$ the Friedmann eq becomes

$$\dot{a} \propto a^{1-n/2} \tag{38}$$

which can be integrated to give for $\rho \propto a^{-n}$

$$a \propto t^{2/n} \tag{39}$$

If we consider a universe dominated by matter, $\Omega = \Omega_M = 1$, $n = 3$ and $\rightarrow$ Einstein-de Sitter model: $a \propto t^{2/3}$. A flat radiation universe (n=4) evolves as $a \propto t^{1/2}$. Although these are models, they are useful to understand the evolution. We expect to have non-zero amounts of radiation, matter and vacuum in a real universe.

We have clear observational evidence that the universe was radiation dominated at early times, matter dominated as it expanded from $a \sim 1/3000$ to $a \sim 1/2$. All the solutions have a singularity at $a = 0$, the Big Bang. Singularity theorems show that any universe with $\rho > 0$ and $p \geqslant 0$ must have begun with a singularity. But we should expect a more complicated theory of gravity to account for the universe at these early times. The case $n = 0$, universe dominated by vacuum energy gives

$$ds^2 = -dt^2 + e^{Ht}(dx^2 + dy^2 + dz^2) \qquad (40)$$

(36) is an example os a S-T with a positive cosmological constant with $k > 0$ and $a \propto \cosh(t/\alpha)$.

How does it compare with (40) where $k = 0$ and $a \propto e^{Ht}$?

It can be shown that calculating the Riemann tensor for (40) it will have the following form:

$$R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \qquad (41)$$

$$\kappa = \frac{R}{n(n-1)} \qquad (42)$$

where $g_\mu\nu$ is a maximally symmetric metric. Since maximally symmetric are the same S-T in different coordinates describing the same manifold or parts of it (in fact (40) only covers part of de Sitter -they are incomplete in the past-).

Another case is an empty universe, $\rho = 0$ with spatial curvature. (18) becomes

$$H^2 = -\frac{\kappa}{a^2} \tag{43}$$

which implies that the curvature must $\kappa < 0$ and we can think that curvature is a fictitious energy density $\rho_C \propto a^{-2}$ which from (39) we see that expands as $a \propto t$: a Milne universe. Using the argumentation above this means that the Milne universe is just a patch of Minkowski in an incomplete coordinate system. (check it: calculate the Riemann tensor...all components vanish). In our real universe we want to parametrize it understanding that radiation density is significantly lower than the density of matter. For convenience we parametrize our universe by Ω_M and Ω_Λ with the curvature fixed by $\Omega_C = 1 - \Omega_M - \Omega_\Lambda$.

Multiple component universes

If we assume a spatially flat universe ($\Omega_C = 0$) we have $\Omega_\Lambda = 1 - \Omega_M$

In general the Friedmann equation can be written

$$H(t)^2 = \frac{8\pi G}{3}\epsilon(t) - \frac{H_0^2}{a^2}(\Omega_0 - 1) \qquad (44)$$

where we $\epsilon(t)$ includes all forms of energy (including vacuum, matter and radiation) and Ω_0 is the dimensionless energy present parameter and the critical density is

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = \epsilon_{c,0} \qquad (45)$$

Current evidence indicates a cosmological constant. We should then consider universes with contributions from matter ($w = 0$), radiation ($w = 1/3$) and a cosmological constant ($w = -1$). (Remember $p = w\rho$, equation (6)).

Then (44) will become

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{R,0}}{a^4} + \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2} \tag{46}$$

$\Omega_{R,0} = \rho_{R,0}/\rho_{c,0}, \Omega_{M,0} = \rho_{M,0}/\rho_{c,0}, \Omega_{\Lambda,0} = \rho_{\Lambda,0}/\rho_{c,0}$, and $\Omega_0 = \Omega_{R,0} + \Omega_{M,0} + \Omega_{\Lambda,0}$.

The "benchmark model" (conforming with current observational) has $\Omega_0 = 1$ and is then spatially flat. But this may not be true.
Multiplying (46) by a^2 we get

$$\frac{\dot{a}}{H_0} = \left[\frac{\Omega_{R,0}}{a^2} + \frac{\Omega_{M,0}}{a} + \Omega_{\Lambda,0} a^2 + 1 - \Omega_0\right]^{1/2} \tag{47}$$

We get

$$H_0 t = \int_0^a \frac{da}{\left[\frac{\Omega_{R,0}}{a^2} + \frac{\Omega_{M,0}}{a} + \Omega_{\Lambda,0} a^2 + 1 - \Omega_0\right]^{1/2}} \tag{48}$$

Equation (48) does not have a simple analytical solution. But it can be integrated numerically for given values of the parameters. One approach is to consider one term dominant over all the other ones. But there are epochs where two of the components are of comparable value.

Matter + curvature

If we consider a universe with only $p = 0 = w$ and spatially flat we already saw that

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{49}$$

In such a universe the temperature decreases monotonically as the universe expands forever ("Big chill"). In a curved one (46)

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1 - \Omega_0}{a^2} \qquad (50)$$

where $\Omega_{M,0} = \Omega_0$ Will this universe expands forever? We know $H_0 > 0$,. To cease expanding $H(t) = 0$. This requires $1 - \Omega_0 < 0$. The universe will cease to expand if $\Omega_0 > 1$, and then $\kappa = +1$.

At the time of maximum expansion $H(t) = 0$ and

$$0 = \frac{\Omega_0}{a^3} + \frac{1 - \Omega_0}{a^2} \tag{51}$$

From where we obtain $a_{max} = \frac{\Omega_0}{\Omega_0 - 1}$ where Ω_0 is the density parameter measure at a scale factor $a = 1$. Mathematically the universe reaches a maximum expansion and then starts contracting until it ends in a Big Crunch. (the contraction is the time reversal and it would require an adiabatic process in which isotropy and homogeneity is maintained). The universe has not only finite spatial extent but also a finite duration in time.

The Matter dominated with $\Omega_0 < 1$ and $\kappa = -1$

In (50) both terms in the right hand side are positive. If the universe is expanding at time t_0 it will keep expanding forever.
At earlier times when $a \ll \Omega_0/(1-\Omega_0)$ matter will dominate and $a \propto t^{2/3}$. Eventually the matter density will diminish and the universe will expand with negative curvature like an empty universe with $a \propto t$. If the universe only contains matter and the

Friedmann equation

$$\frac{\dot{a}^2}{H_0^2} = \frac{\Omega_0}{a} + (1 - \Omega_0) \tag{52}$$

we get

$$H_0 t = \int_0^a \frac{da}{\left[\frac{\Omega_0}{a} + (1 - \Omega_0)\right]^{1/2}} \tag{53}$$

When $\Omega_0 \neq 1$ we get the solution in parametric form.

If $\Omega_0 > 1$

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{(\Omega_0 - 1)}(1 - \cos\theta) \tag{54}$$

and

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta) \tag{55}$$

with $0 < \theta < 2\pi$. The time between the Big Bang at $\theta = 0$ and the time at the Big Crunch ($\theta = 2\pi$):

$$t_{crunch} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{56}$$

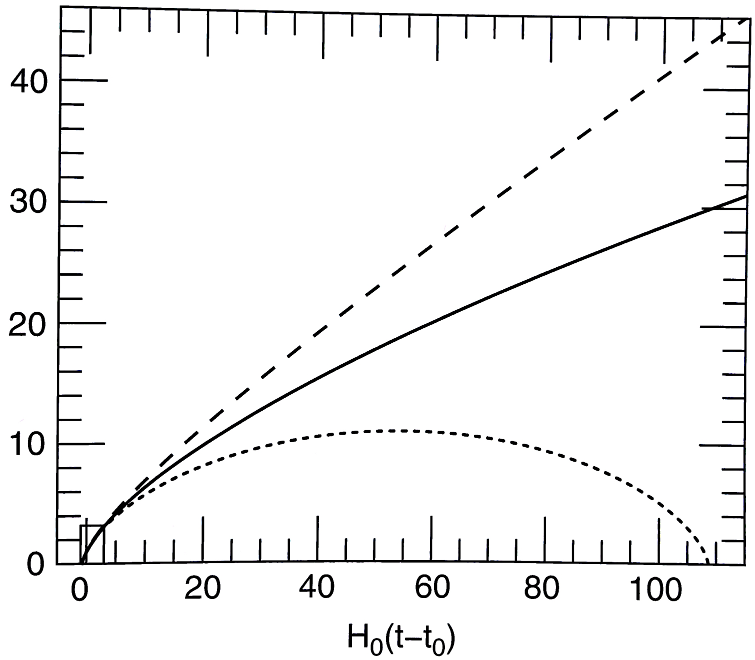

Fig 1 Scale factor with matter: Solid line $\Omega_0 = 1$, dashed $\Omega_0 = 0.9$, dotted $\Omega_0 = 1.1$

Credit: Ryden 2017

In the case of $\Omega_0 < 1$ the solution of (53) can be written

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)}(\cosh\eta - 1) \tag{57}$$

and

$$t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\eta - \eta) \tag{58}$$

with $0 < \eta < \infty$ Notice in Fig 1 how hard is to distinguish the three models at $t = t_0$.

Matter + Cosmological constant

Let's consider a universe spatially flat with a cosmological constant. At $t = t_0$ for the space to be flat we need

$$\Omega_{\Lambda,0} + \Omega_{M,0} = 1 \tag{59}$$

and the Friedmann equation

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0} \tag{60}$$

where due to (59) it ends up being

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{M,0}}{a^3} + (1 - \Omega_{M,0}) \tag{61}$$

The second term represents the contribution of the cosmological constant, the first is always positive because it represents the matter contribution.

The second term will be positive if

$\Omega_{M,0} < 1$ ($\Omega_{\Lambda,0} > 0$) and negative if
$\Omega_{M,0} > 1$ ($\Omega_{\Lambda,0} < 0$).
A flat universe with $\Omega_{\Lambda,0} > 0$ will continue to expand forever if it's expanding at $t = t_0$ (Big Chill). But if $\Omega_{\Lambda,0} < 0$ the negative value provides an attractive force. The universe will cease to expand at

$$a_{max} = \left(\frac{\Omega_{M,0}}{\Omega_{M,0} - 1} \right)^{1/3} \tag{62}$$

and will collapse back down to $a = 0$ (Big Crunch) at

$$t_{crunch} = \frac{2\pi}{3H_0} \frac{1}{\sqrt{\Omega_{M,0} - 1}}. \tag{63}$$

For a fixed value of H_0 the larger the value of $\Omega_{M,0}$ the shorter the lifetime of the universe.

For a flat $\Omega_{\Lambda,0} < 0$ universe the Friedmann equation can be integrated to give

$$H_0 t = \frac{2}{3\sqrt{\Omega_{M,0} - 1}} \sin^{-1}\left[\left(\frac{a}{a_{max}}\right)^{3/2}\right] \qquad (64)$$

A flat universe with a $\Omega_{\Lambda,0} < 0$ ends in a big crunch like the positively curved matter-only universe. For comparison: a $\kappa > 0$ $\Omega_{M,0} = 1.1$ gets to the BC in $t_{crunch} \approx 110 H_0^{-1}$. A flat one with also $\Omega_{M,0} = 1.1$ but $\Omega_{\Lambda,0} = -0.1$ lives only $t_{crunch} \approx 7 H_0^{-1}$.

In a flat universe with $\Omega_{M,0} < 1$ and $\Omega_{\Lambda,0} > 0$ both density contributions are equal at the scale factor

$$a_{M\Lambda} = \left(\frac{\Omega_{M,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{M,0}}{1 - \Omega_{M,0}}\right)^{1/3} \qquad (65)$$

For a flat $\Omega_{\Lambda,0} > 0$ universe the Friedmann equation can be integrated

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{M,0}}} \ln \left[\left(\frac{a}{a_{M\Lambda}} \right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{M\Lambda}} \right)^3} \right] \tag{66}$$

At early times when $a \ll a_{M\Lambda}$ eq (66) reduces to

$$a(t) \approx \left(\frac{3}{2} \sqrt{\Omega_{M0}} H_0 t \right)^{2/3} \tag{67}$$

Notice the $t^{2/3}$ dependence. At late times when $a \gg a_{M\Lambda}$, (66) yields

$$a(t) \approx a_{M\Lambda} \exp(\sqrt{1-\Omega_{M0}} H_0 t). \tag{68}$$

Notice the $a \propto e^{Kt}$ dependence of a flat dominated universe.

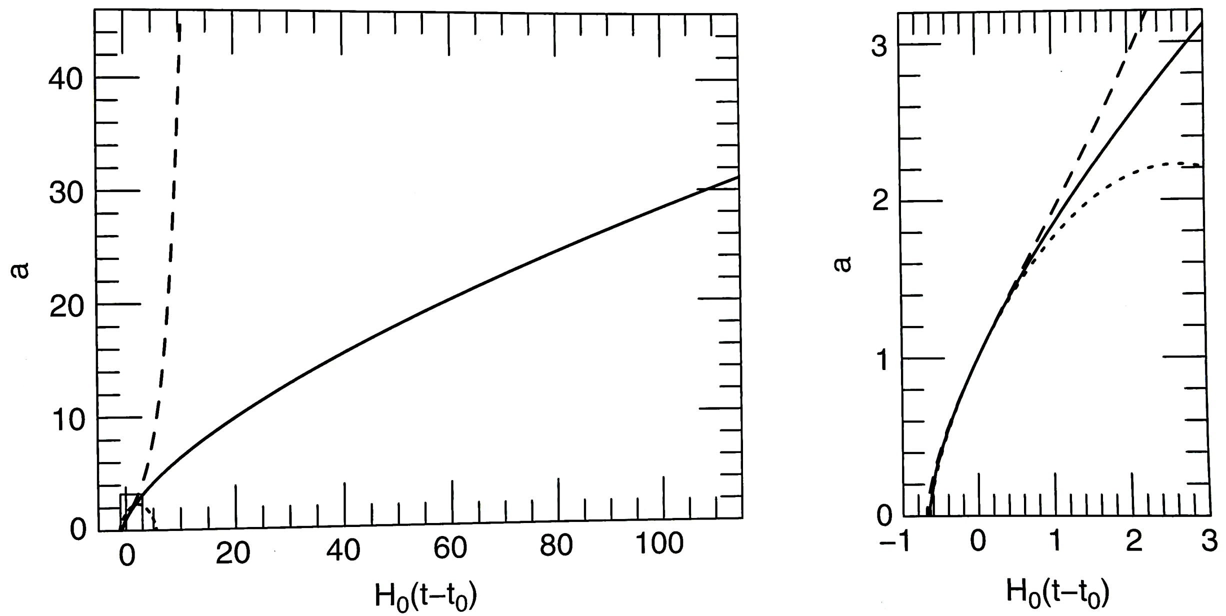

Fig 2 Scale factor with matter and lambda: Solid line $\Omega_{M,0} = 1$, $\Omega_{\Lambda,0} = 0$ dashed $\Omega_{M,0} = 0.9$, $\Omega_{\Lambda,0} = 0.1$, dotted $\Omega_{M,0} = 1.1$, $\Omega_{\Lambda,0} = -0.1$. Right panel early times.

Credit: Ryden 2017

If we are in a flat universe with only matter and cosmological constant, and we measure H_0 and Ω_{M0} (66) would give

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{M,0}}} \ln\left[\frac{\sqrt{1-\Omega_{M,0}}+1}{\sqrt{\Omega_{M,0}}}\right]. \qquad (69)$$

If we use the values $H_0 = 68 \pm 2 km/sMpc$ and $\Omega_{M0} = 0.31$ and $\Omega_{\Lambda,0} = 0.69$ (benchmark model) we find that

$$t_0 = 0.955 H_0^{-1} = 13.74 \pm 0.40 \text{ G}yr \qquad (70)$$

If the universe is well described by the benchmark model the cosmological constant has been a dominant component of the universe for the last 3.6 billion years.

Matter + Curvature + Lambda
Choosing different $\Omega_{\Lambda,0}$ and Ω_{M0} and allowing different curvature values we can investigate some interesting models.

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_{M0} - \Omega_{\Lambda 0}}{a^2} \qquad (71)$$

If both $\Omega_{\Lambda,0} > 0$ and $\Omega_{M0} > 0$ the first and the third term of the right hand side are positive.

But if $\Omega_{\Lambda,0} + \Omega_{M0} > 1$ (universe with $\kappa > 0$) then the second term is not.

This would give some range of values for H^2 negative, an unphysical result.

Imagine such a universe starts with $a \gg 1$ and $H < 0$, i.e. contracting from a low-density Λ dominated state, the negative curvature term in (71) becomes dominant causing the contraction to stop ar a minimum $a = a_{min}$, and then expand again in a big bounce.

This indicates it is possible (mathematically) to have a universe that expands outward at a later time but never had a Big-Bang ($a = 0$ at $t = 0$).

Another possibility is a "loitering" universe. It starts matter dominated, expands with $a \propto t^{2/3}$, then for a long range of time a is constant. Until the cosmological constant takes over and the universe starts to expand exponentially.

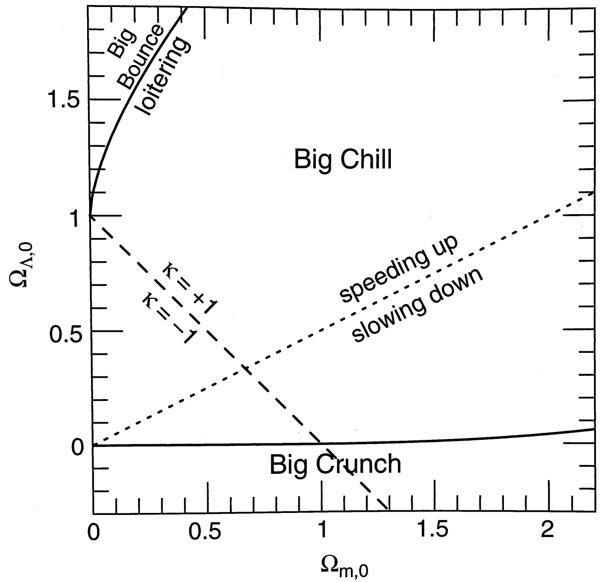

Fig 3 dash line $\kappa = 0$ Dotted lines indicates universes that are not accelerating $q_0 = 0$ at $a = 1$. Regions where the big chill is possible ($a \to \infty$ as $t \to \infty$), regions of big crunch ($a \to 0$ as $t \to t_{crunch}$), a loitering phase ($a \approx constant$for a long time) or a Big Bounce ($a = a_{min} > 0$ at $t = t_{bounce}$).

Credit: Ryden 2017

In the region labeled “Big Crunch” the universe starts with $a = 0$ at $t = 0$ reaches a maximum a_{max}, the re-collapses to $a = 0$ at a time $t = t_{crunch}$.

Big crunch universes can have all 3 types of curvature. In the region Big-Chill the universe starts with $a = 0$ at $t = 0$ and expands forever. Same thing with the curvatures.
In the region Big Bounce the universe starts in a contracting state reaches a minimum $a = a_{min} > 0$ and then expands for ever.

Universes that fall just below the dividing line between Big Bounce and Big Chill are loitering universes. The closer it gets to that line the longer the loitering state.

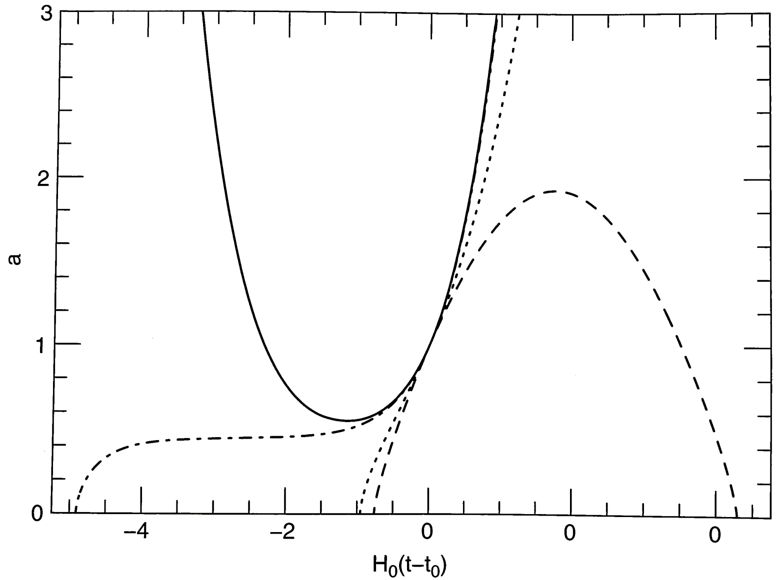

Fig 4 All four universes have $\Omega_{M0} = 0.31$. Dotted line a Big Chill with ($\Omega_{\Lambda 0} = 0.69$, $\kappa = 0$), Dashed line a Big Crunch ($\Omega_{\Lambda 0} = -0.31$, $\kappa = -1$). Dot-dashed line a loitering universe ($\Omega_{\Lambda 0} = 1.7289$, $\kappa = +1$). Solid line a Big Bounce ($\Omega_{\Lambda 0} = 1.8$, $\kappa = +1$)

Credit: Ryden 2017

Radiation + matter

In our universe radiation and matter equal each other at $a_{RM} \equiv \Omega_{R0}/\Omega_{M0} \approx 2.9 \times 10^{-4}$. At scale factors $a \ll a_{RM}$ the universe is well described by a flat, radiation-only model. At scale factors $a \sim a_{RM}$ a flat model with both radiation and matter make a better fit.

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{R,0}}{a^4} + \frac{\Omega_{M,0}}{a^3} \tag{72}$$

which can be rearranged

$$H_0 dt = \frac{a da}{\Omega_{R0}^{1/2}} \left[1 + \frac{a}{a_{RM}}\right]^{-1/2} \tag{73}$$

Integration gives

$$H_0 t = \frac{4a_{RM}^2}{3\sqrt{\Omega_{R0}}}\left[1 - \left(1 - \frac{a}{2a_{RM}}\right)\left(1 + \frac{a}{a_{RM}}\right)^{1/2}\right] \quad (74)$$

In the limit $a \ll a_{RM}$ this gives

$$a \approx (2\sqrt{\Omega_{R0}}H_0 t)^{1/2}, \quad (75)$$

which is the expected result for a radiation dominated phase of the evolution.
For $a \gg a_{RM}$ and before Λ or curvature contributes significantly in the Friedmann equation

$$a \approx \left(\frac{3}{2}\sqrt{\Omega_{M0}}H_0 t\right)^{2/3}, \quad (76)$$

The time when radiation is equal to matter can be obtained setting $a = a_{RM}$ in (74)

$$t_{RM} = \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{a_{RM}^2}{\sqrt{\Omega_{R0}}}H_0^{-1} \approx 0.391\frac{\Omega_{R0}^{3/2}}{\Omega_{M0}^2}H_0^{-1}. \quad (77)$$

For the Benchmark model the epoch when the universe was balanced in radiation and matter, we use $\Omega_{R0} = 9.0 \times 10^{-5}$, $\Omega_{M0} = 0.31$ and $H_0^{-1} = 14.4$ Gyr, then

$$t_{RM} = 3.47 \times 10^{-6} H_0^{-1} = 50000 \text{ yr}. \quad (78)$$

The epoch when the universe was radiation dominated was only 50,000 years long (z= 4,600). The decoupling took place about 380,000 years after the big Bang (z=1000).

The Benchmark Model

This is the model that fits observational data: it is spatially flat, and contains matter, radiation and a cosmological constant. The Hubble constant is taking to be $H_0 = 68 km/s/Mpc$. The radiation consist of photons and neutrinos. The photons are assumed to be provided by cosmic microwave background with current temperature $T_0 = 2.7255$ K and density parameter $\Omega_{\gamma 0} = 5.35 \times 10^{-5}$. The energy density of neutrinos is calculated theoretically to be 68.1% of the of the CMB (assuming relativistic neutrinos).

Photons:	$\Omega_{\gamma 0} = 5.35 \times 10^{-5}$
Neutrinos:	$\Omega_{\nu 0} = 3.65 \times 10^{-5}$
Total radiation:	$\Omega_{R0} = 9.00 \times 10^{-5}$
Baryonic matter:	$\Omega_{Bar0} = 0.048$
Nonbaryonic dark matter:	$\Omega_{DM0} = 0.262$
Total matter:	$\Omega_{M0} = 0.31$
Cosmological constant	$\Omega_{\Lambda 0} = 0.69$

Important epochs

Radiation-matter equality:	$a_{RM} = 2.9 \times 10^{-4}$	$t_{RM} = 0.050\text{Myr}$
Matter-Lambda equality:	$a_{\Lambda M} = 0.77$	$t_{\Lambda M} = 10.2\text{Gyr}$
Now:	$a_0 = 1$	$t_0 = 13.7\text{Gyr}$

With Ω_{R0}, Ω_{M0} and $\Omega_{\Lambda 0}$ known, the scale factor can be computed numerically using the Friedmann equation in the form (46)

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{R,0}}{a^4} + \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2} \qquad (46)$$

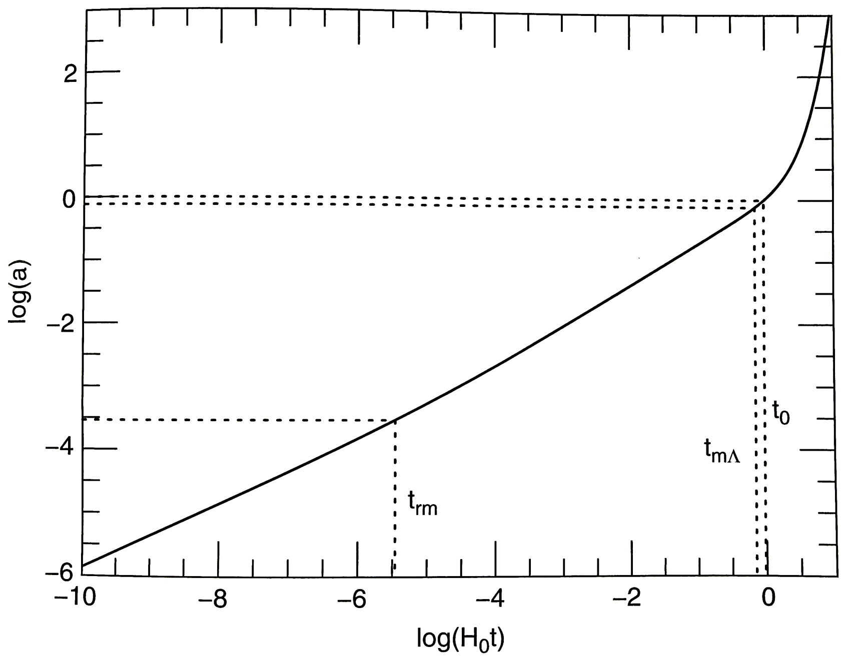

Fig 5 The scale factor as a function of time (in units of the Hubble time). The dotted lines indicate the time of the radiation-matter equality, $a_{RM} = 2.9 \times 10^{-4}$, the time of matter and Λ equality, $a_{M\Lambda} = 0.77$ and the present time, $a_0 = 1$. *Credit: Ryden 2017*

Once we have calculated $a(t)$ other properties of the Benchmark model can be computed.

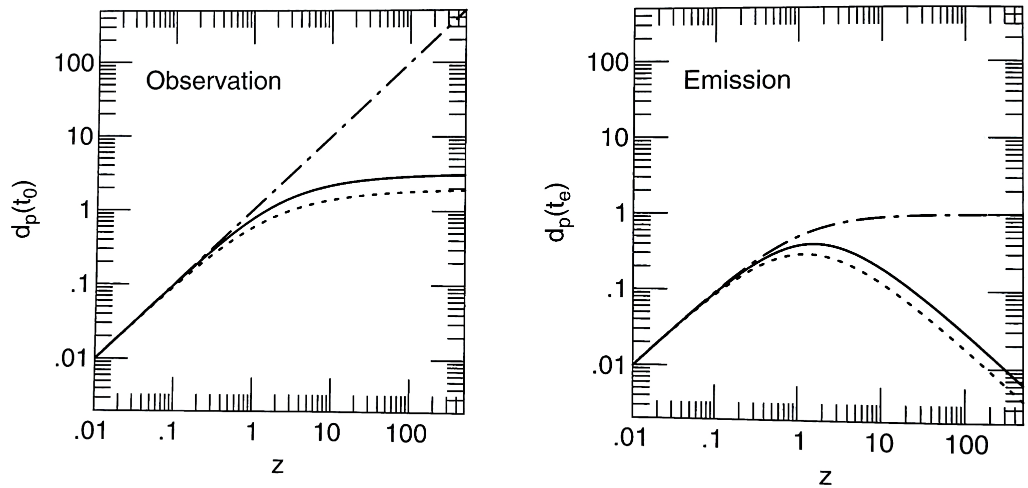

Fig 6 The proper distance to a light source with z, in units of the Hubble distance.Left: d_p at time of observation, Right: d_p at time of emission. Solid line is the Benchmark model. Dot-Dashed is a flat, Λ only model and the dotted line a flat, Matter only universe. *Credit: Ryden 2017*

In the limit $z \to \infty$, the proper distance $d_p(t_0)$ approaches a limiting value $d_p \to 3.20c/H_0$ for the Benchmark model.

The benchmark model has then a finite horizon distance

$$d_{hor}(t_0) = 3.20c/H_0 = 3.35ct_0 = 14000\text{Mpc} \tag{79}$$

If the Benchmark model is the correct one this means we can not see objects farther than 14 gigaparsecs away because light from them has not yet had time to reach us.

In Figure 6 we have plotted the distance $d_p(t_e)$, the distance to a galaxy with observed redshift z at the time the observed photons were emitted. In this model $d_p(t_e)$ has a maximum for galaxies with $z = 1.6$, where $d_p(t_e)0.405c/H_0$.
Regarding how long has the light have been traveling? computing the lookback time. If light emitted at time t_e is observed at time t_o the lookback time is $t_0 - t_e$. In the limit of small zs

$t_0 - t_e \approx z/H_0$. But in figure 7 we can see that the relationship between lookback time and z becomes non-lineal.

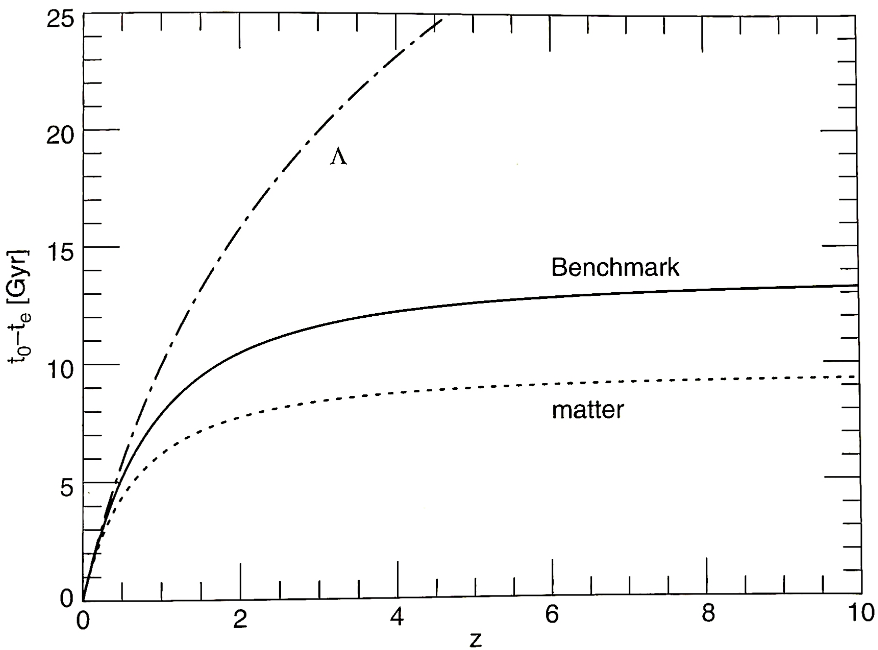

Fig 7 $t_0 - t_e$ for galaxies with observed z. Hubble time is $H_0^{-1} = 14.4\text{Gyr}$. *Credit: Ryden 2017*

If we consider a galaxy with $z = 2$ in the Benchmark model $t_0 - t_e = 10.5\text{Gyr}$ we are seeing a redshifted image of that galaxy as it was 10.5 billion years ago. In a flat, Λ only universe, on the other hand the lookback time would be 15.8 billion years ago for an $H_0^{-1} = 14.4\text{Gyr}$.
For a flat, matter dominated universe, it would be only 7.7 Gyr.
As of early 2026, the highest redshift galaxy definitively observed is MoM-z14 at a redshift of approximately $z = 14.44$, discovered by the James Webb Space Telescope (JWST). It represents the universe as it was 280 million years after the Big Bang.