Lesson 6
At last!
the Exodus from Newton's world

Mario Díaz

The University of Texas Rio Grande Valley

mario.diaz@utrgv.edu

February 7, 2026

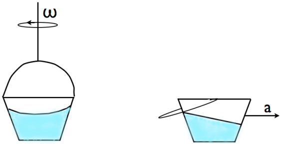
Historically Einstein was guided by 5 basic principles or ideas:

- Mach's principle.
- principle of equivalence.
- principle of covariance.
- principle of minimal gravitational coupling.
- correspondence principle.

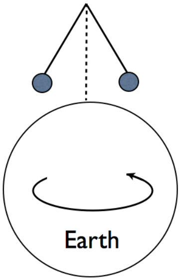
Regarding Mach's principle Einstein called it a principle for the first time.
Mach laid out its fundamentals:
"[The] investigator must feel the need of... knowledge of the immediate connections, say, of the masses of the universe. There will hover before him as an ideal insight into the principles of the whole matter, from which accelerated and inertial motions will result in the same way."

Mach's principle

Mach was trying to demolish Newton's concept of absolute space. A crucially related concept is that of non-inertial frames. Inertial frames are defined as privileged ones in an absolute space. What are Nature law's in non-inertial frames? How inertial frames came to be? The forces that "appear" in non-inertial frames are called inertial forces (i.e. a rotating system or an accelerated system of reference). Are these fictitious? What is the origin of these forces? Newton postulated the existence of an absolute space. The bucket can elucidate when a system is inertial.



For Mach it was the existence of all the masses in the universe, the fixed stars beyond the earth, which determine a local inertial frame. It is a system in a privileged state of motion with respect to the stars. The inertial forces are related to matter of which the stars are made of. Just look at a pendulum in the North pole. For Newton this is a non rotating frame (relative to absolute space). But for an observer on earth the pendulum swings 360° in one day exactly. The corolary from Mach's principle is that inertial frames are those in which the fixed stars are not rotating.



Should there be an anisotropy in the distribution of matter in the universe we would detect it in the fact that inertial forces would not be isotropic. Hughes and Drever run an experiment detecting anisotropy null to 10^{-20}.

Hughes et al., Phys. Rev. lett. 4 no. 1 (1960), pg 342.

Drever, Philosophical Mag. 6, 683.

This extremely accurate experiment looked for any anisotropy in nuclear magnetic resonance. Hughes placed an upper limit on such anisotropy of 10^{-20}.

There were variations on the Hughes-Drever experiment:

Prestage et al., Physics Review Letters 54, 2387 (1985).

Lamoreaux et al., Physics Review Letters 57, 3125 (1986).

Chupp et al., Phys. Rev. Lett. 63, 1541 (1989).

Other tests:

Phillips, Phys. Rev. Lett. 59 no. 15 (1987), pg 1784.

A test using a cryogenic torsion pendulum carrying a transversely polarized magnet. No significant anisotropy was observed.

Hou, L.-S., Ni, W.-T., and Li, Y.-C.M., ÒTest of Cosmic Spatial Isotropy for Polarized Electrons Using a Rotatable Torsion BalanceÓ, Phys. Rev. Lett., 90, 201101, (2003).

Heckel et al., Phys. Rev. Lett. 97 (2006) 021603. arXiv:hep-ph/0606218

Here is a summary of Machian ideas (from d'Inverno):

- The matter distribution determines the geometry.
- If there is no matter then there is no geometry.
- A body in an empty universe should have no inertial properties.

Newtonian equivalence of mass

We have discussed before the equivalence of inertial mass and gravitational mass.
We could have actually differentiate three type of masses:

- inertial mass.
- passive gravitational mass.
- active gravitational mass.

All experiments show that these three are identical up to 10^{-12}.

The principle of equivalence

There are different versions, stronger or weaker, of the principle:

- The motion of a gravitational test particle in a gravitational field is independent of its mass and it composition.
- The gravitational field is coupled to everything.
- There are no local experiments which can distinguish non-rotating free fall in a gravitational field from uniform motion in space in the absence of a gravitational field.
- A frame linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field.

Notice that we can formulate this mathematically in the following language: A test particle in Minkowski moves according to:

$$\frac{d^2x^a}{d\tau^2} = 0. \tag{1}$$

In a noninertial system of reference:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0. \tag{2}$$

Notice that if want to regard $\Gamma^a{}_{bc}$ as force terms, then g_{ab} has to be seen as potentials.
We need to generalize these ideas to build a relativistic theory of gravitation.

Einstein propose then the following two principles to build a theory of gravitation consistent with relativity:

- *Principle of General Relativity*
  All observers are equivalent. In special relativity we have preferred systems, Minkowski coordinates. In a general curved space time we don't have a preferred coordinate system. (although there are symmetries).
- *Principle of General Covariance*
  The equations of physics should have tensorial form. What this means is that the theory should be invariant under coordinate transformations.

The principle of minimal gravitational coupling

This is a simplicity principle when making the transition to general relativity from special relativity. i.e. if we have the conservation law:

$$\partial_b T^{ab} = 0 \tag{3}$$

The simplest generalization is:

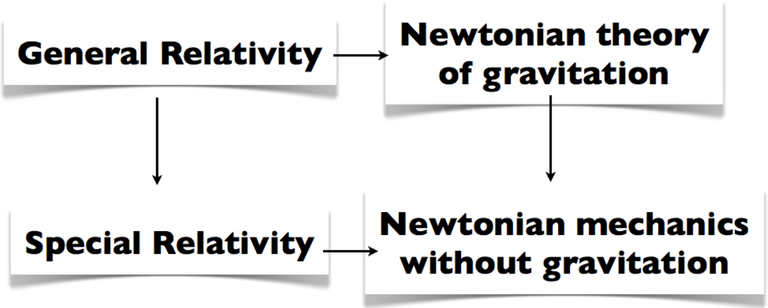
$$\nabla_b T^{ab} = 0 \tag{4}$$

We could have taken:

$$\nabla_b T^{ab} + g^{be} R^a{}_{bcd} \nabla_e T^{cd} = 0 \tag{5}$$

but the Principle would stay: "No terms explicitly containing the curvature tensor should be added when making the transition."

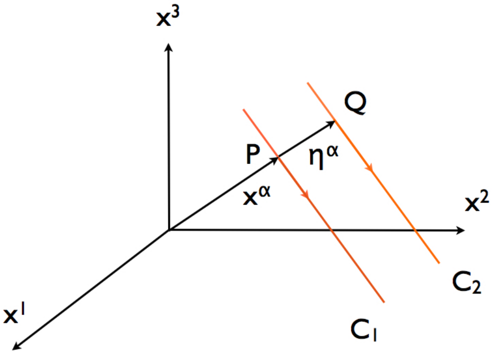
The correspondence principle



The Newtonian equation of deviation

(following D'Inverno)

The metric is $ds^2 = dx^2 + dy^2 + dz^2$.
Let's look at the paths of two neighboring particles acted upon by a gravitational field whose potential is ϕ:



For the particle traveling on C_1 the parametric equations are:

$$x^\alpha = x^\alpha(t) \tag{6}$$

And for the one on C_2:

$$x^\alpha = x^\alpha(t) + \eta^\alpha(t) \tag{7}$$

Then the equations of motion are:

$$\ddot{x}^\alpha = -(\partial^\alpha \phi)_P \tag{8}$$

For the second one:

$$\ddot{x}^\alpha + \ddot{\eta}^\alpha = -(\partial^\alpha \phi)_Q \tag{9}$$

We do a Taylor expansion:

$$-(\partial^\alpha \phi)_Q = -(\partial^\alpha \phi)_P - (\eta^\beta \partial_\beta \partial^\alpha \phi)_P \tag{10}$$

Subtracting (8) from (9):

$$\ddot{\eta}^\alpha = -\eta^\beta \partial_\beta \partial^\alpha \phi \tag{11}$$

Defining:

$$K^\alpha{}_\beta = \partial_\beta \partial^\alpha \phi \tag{12}$$

the newtonian equation of deviation becomes:

$$\ddot{\eta}^\alpha + K^\alpha{}_\beta \eta^\beta = 0 \tag{13}$$

It is not difficult to see that Laplace equations can be written:

$$K^{\alpha}{}_{\alpha} = 0 \tag{14}$$

We can compare now equation (13) with (62) from Lesson 7:

$$\nabla_V(\nabla_V \xi^a) - R^a{}_{bcd} V^b V^c \xi^d = 0 \tag{15}$$

$$\frac{D^2 \xi^a}{d\tau^2} - R^a{}_{bcd} V^b V^c \xi^d = 0 \tag{16}$$

where ξ is η and V is x.

The vacuum field equations of general relativity

What would be the equivalent of the $K^{\alpha}_{\alpha} = 0$ in the newtonian theory?
We need to define a frame or tetrad:
We do so by taking the three unit spacelike cartesian vectors and pick a third one, timelike:

$$e_0{}^a \equiv v^a, \tag{17}$$

We can summarize the orthogonality of this tetrad:

$$e_i{}^a e_{j\,a} = \eta_{ij} \tag{18}$$

where η_{ij} is the Minkowski metric. We can now define a connecting vector whose spatial frame components are:

$$\eta^{\alpha} = e^{\alpha}_a \eta^{\alpha} \tag{19}$$

The analogous equation in a curved space time to the one we found for "newtonian deviation" would be:

$$\frac{D^2\eta^\alpha}{D\tau} + K^\alpha{}_\beta \eta^\beta = 0 \tag{20}$$

where:

$$K^\alpha{}_\beta \eta^\beta = -R^a{}_{bcd} e^\alpha{}_a v^b v^c e_\beta{}^d. \tag{21}$$

From these we could infer how to write the **vacuum field equations**.
The analogous to the vanishing of the trace is:

$$R^a{}_{bcd} e^\alpha{}_a v^b v^c e_\beta{}^d = 0. \tag{22}$$

We can investigate the meaning of this by using a coordinate system:

$$e_i{}^a = \delta_i{}^a \tag{23}$$

Then (22) reduces to:

$$R^\alpha{}_{00\alpha} = 0 \tag{24}$$

(where α runs from 1 to 3 only.) And due to the antisymmetry in the last pair of indices:

$$R^0{}_{000} = 0 \tag{25}$$

These can be combined in :

$$R^{a}{}_{00a} = 0 \tag{26}$$

$$R^{a}{}_{00a} = 0 = R^{a}{}_{bca} v^{b} v^{c} = -R^{a}{}_{bac} v^{b} v^{c} = -R_{bc} v^{b} v^{c} \tag{27}$$

this means:

$$R_{ab} = 0 \tag{28}$$

But this is equivalent to the vanishing of the Einstein tensor:

$$G_{ab} = 0 \tag{29}$$

These are the equations that Einstein proposed to serve as the vacuum field equations of general relativity.

The full field equations

When we have other fields we can encode their information in the energy-momentum tensor T^{ab}. The equivalence of mass and energy suggest that all forms of energy act as sources for the gravitational field. If we assume that T^{ab} be the source of the field equations we know that:

$$\partial_b T^{ab} = 0 \tag{30}$$

This can be generalized to:

$$\nabla_b T^{ab} = 0 \tag{31}$$

But we also know that the EInstein tensor satisfies the Bianchi identities:

$$\nabla_b G^{ab} \equiv 0 \tag{32}$$

The last two equations suggest that the two tensors are proportional to one another. We can propose:

$$G^{ab} = \kappa T^{ab} \tag{33}$$

In no relativistic units we will see that κ following the correspondence principle:

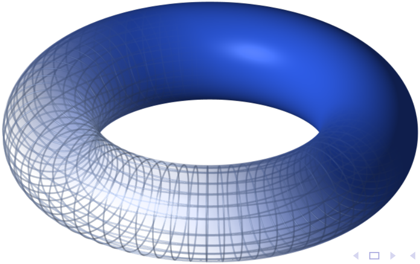
$$\kappa = 8\pi G/c^4 \tag{34}$$

So the full GR equations are:

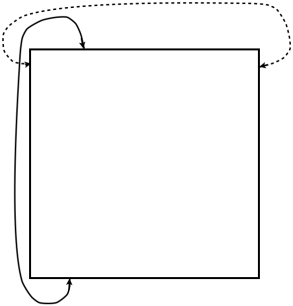
$$G^{ab} = \frac{8\pi G}{c^4} T^{ab} \tag{35}$$

adapted from Sean Carroll

If we compare a straight line and a circle we would say that the first one has no curvature but that the second one is curved, but after our definition of the Riemann tensor this can not be true. The Riemmann tensor is identically zero in one dimension. Our intuitive definition of curvature is always thinking about an embedding in Euclidean space. Let's discuss the following example: A torus is an object in two dimensions where the Riemann tensor has only one independent component.



A torus can be considered as a square region of the plane with opposite sides identified.



A valid metric for the torus could be the following one with $0 < u, v < 2\pi$

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 \sin^2 dv^2 \qquad (36)$$

The following transformation

$$x = (c + a\cos v)\cos u \tag{37}$$
$$y = (c + a\cos v)\sin u \tag{38}$$

could change it into $ds^2 = dx^2 + dy^2$.

On the other hand let's look at the sphere S^2:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{39}$$

The non=zero connection coefficients for (39) are:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \tag{40}$$
$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta \tag{41}$$

And the components of the Riemann tensor:

$$R^{\theta}_{\phi\theta\phi} = \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\theta\phi} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi} \tag{42}$$

$$=(\sin^2\theta - \cos^2\theta) - 0 + 0 - (-\sin\theta\cos\theta)(\cot\theta) \tag{43}$$

$$=\sin^2\theta \tag{44}$$

lowering the index we get:

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda} R^{\lambda}\phi\theta\phi \tag{45}$$

$$= g_{\theta\theta} R^{\theta}\phi\theta\phi \tag{46}$$

$$= a^2 \sin^2\theta \tag{47}$$

Computing the Ricci tensor $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$:

$$R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1 \tag{48}$$

$$R_{\theta\phi} = R_{\phi\theta} = 0 \tag{49}$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta \tag{50}$$

And the Ricci scalar is:

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}. \tag{51}$$

adapted from Schutz

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2). \tag{52}$$

We expect that far from the source $\phi = -GM/r$. Also we assume in all this that $|m\phi| \ll m$. We can compute the motion of a freely falling particle. If $\vec{p} = m\vec{U}$, where $\vec{U} = d\vec{x}/d\tau$ we will have:

$$\nabla_{\vec{U}}\vec{U} = 0 \tag{53}$$

If the proper time τ is the affine parameter along the geodesic so it is τ/m and we have:

$$\nabla_{\vec{p}}\vec{p} = 0 \tag{54}$$

which is also good for photons.

Let's look at the zero component:

$$m\frac{d}{d\tau}p^0 + \Gamma^0{}_{\alpha\beta}p^\alpha p^\beta = 0 \tag{55}$$

The particle is moving with $v \ll c$ so:

$$m\frac{d}{d\tau}p^0 + \Gamma^0{}_{00}(p^0)^2 = 0 \tag{56}$$

and

$$\Gamma^0{}_{00} = \phi_{,0} + O(\phi^2) \tag{57}$$

we get:

$$\frac{d}{d\tau}p^0 = -m\frac{\partial\phi}{\partial\tau} \tag{58}$$

The space components give:

$$p^\alpha p^i{}_{,\alpha} + \Gamma^i{}_{\alpha\beta} p^\alpha p^\beta = 0 \tag{59}$$

$$m\frac{dp^i}{d\tau} + \Gamma^i{}_{00}(p^0)^2 = 0 \tag{60}$$

which gives:

$$\frac{dp^i}{d\tau} = -m\phi_{,j}\delta^{ij}. \tag{61}$$

We can revisit the Exercise 1 from the last test, but this time with a general metric: The geodesic equation can be written for $\vec{p}$

$$p^\alpha p_{\beta;\alpha} = 0 \tag{62}$$

this would give:

$$m\frac{dp_\beta}{d\tau} = \Gamma^\gamma{}_{\beta\alpha} p^\alpha p_\gamma \tag{63}$$

Which can also be written in terms of the metric:

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p_\nu p^\alpha \tag{64}$$

But if all the components of the metric are independent of x^β for all β, then p_β is a constant along the particle's trajectory.
Notice that if the metric does not depend on the time, we can find a coordinate system in which the metric components are time independent and p^0 will be conserved (this is the energy).
If we apply metric (36) to the definition of momentum we get:

$$\vec{p}\cdot\vec{p} = -m^2 = g_{\alpha\beta} p^\alpha p^\beta \tag{65}$$

$$= -(1+2\phi)(p^0)^2 + (1-2\phi)[(p^x)^2 + (p^y)^2 + (p^z)^2], \tag{66}$$

We can solve for p^0:

$$(p^0)^2 = [m^2 + (1-2\phi)(p^2)](1+2\phi)^{-1}, \tag{67}$$

where p^2 refers to the space coordinates. And we still assume $|\phi| \ll 1, |\mathrm{p}| \ll m$, so:

$$\begin{aligned}
(p^0)^2 &= m^2(1-2\phi) + (1-2\phi)(1-2\phi)p^2 && (68)\\
&\approx m^2 - 2\phi m^2 + p^2 && (69)\\
p^0 &\approx m(1 - 2\phi + \frac{p^2}{m^2})^{1/2} && (70)\\
&\approx m(1 - \phi + \frac{p^2}{m^2}) && (71)
\end{aligned}$$

We can lower the index:

$$p_0 = g_{0\alpha} p^\alpha = -(1+2\phi)p^0 \tag{72}$$

$$= -(1+2\phi)m(1-\phi+\frac{p^2}{m^2}) \tag{73}$$

$$-p_0 = m + m\phi + p^2/2m \tag{74}$$

The terms are the rest mass, the potential energy and the kinetic energy.

A general gravitational field will not be stationary in *any* frame, which means that we can not define a globally conserved energy.

We can now try looking at the result of the metric being axially symmetric (let's say it does not depend of an angle ψ):

$$p_\psi = g_{\psi\psi} p^\psi \approx g_{\psi\psi} m d\psi/dt \approx m g_{\psi\psi} \Omega, \tag{75}$$

where Ω is the angular velocity of the particle. For a nearly flat metric,

$$g_{\psi\psi} = \vec{e}_\psi \cdot \vec{e}_\psi \approx r^2 \tag{76}$$

in cylindrical coordinates r, ψ, z so the conserved quantity is:

$$p_\psi \approx m r^2 \Omega \tag{77}$$

More about symmetries:
Let's try to make this notion of symmetry more rigorous.
Symmetries of the metric are called isometries. For example $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$ has several isometries. These include translations:

$$x^\mu \to x^\mu + a^\mu \tag{78}$$

and Lorentz transformations:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{79}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz-transformation matrix. These are a total of ten isometries. We already saw that if

$$\partial_{x^{\sigma*}} g_{\mu\nu} = 0 \Rightarrow x^{\sigma*} \to x^{\sigma*} + a^{\sigma*} \tag{80}$$

is a symmetry.

In particular we already saw that the geodesic equation will yield if:

$$\partial_{x^{\sigma *}} g_{\mu\nu} = 0 \Rightarrow \frac{dp_{\sigma *}}{d\tau} = 0 \tag{81}$$

In general isometries are more rigorously defined in terms of the Lie derivative of the metric:

$$\partial_{x^{\sigma *}} g_{\mu\nu} = 0 \Rightarrow \frac{dp_{\sigma *}}{d\tau} = 0 \tag{82}$$

$$\mathcal{L}_{\vec{V}} g_{\mu\nu} = V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} \tag{83}$$

$$= \nabla_\mu V_\nu + \nabla_\nu V_\mu, \tag{84}$$

where we calculate $\mathcal{L}_{\vec{V}} g_{\mu\nu}$ from definitions (58) and (59) in Lesson 7.

In short:

$$\mathcal{L}_{\vec{V}} g_{\mu\nu} = \nabla_{(\mu} V_{\nu)}, \tag{85}$$

where ∇_μ is the covariant derivative calculated using $g_{\mu\nu}$. If

$$\mathcal{L}_{\vec{V}} g_{\mu\nu} = 0, \tag{86}$$

then $\vec{V}$ is a Killing vector of the metric and it defines one isometry of the metric. i.e. for the Minkowski metric in standard Cartesian coordinates $\partial/\partial_{x^\alpha}$ where $\alpha = t, x, y, z$ are Killing vectors. From these we can see that Killing vector fields define geodesics along which the corresponding momentum is a conserved quantity:

$$\nabla_{(\mu} V_{\nu)} = 0 \Rightarrow p^\mu \nabla_\mu (V_\nu p^\nu) = 0. \tag{87}$$

In an arbitrary spacetime manifold (not necessarily homogeneous or isotropic) we can do the following:

1. pick an initial spacelike hypersurface S_I,
2. place an arbitrary coordinate grid (x^1, x^2, x^3) on it,
3. look at the geodesic world lines orthogonal to it and attach to them:
4. coordinates $(x^1, x^2, x^3) = constant$, $x^0 \equiv t = t_I + \tau$ where τ is the proper time along the world line, with $\tau_{S_I} = 0$.

Now if we have a general $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ since $x^i = constant$ along the geodesics then $ds^2 = g_{00} dt^2$ along the geodesics.
But along the geodesics $ds^2 = -d\tau^2$ so $g_{00} = -1$ everywhere.

Let now $\vec{e}_\alpha$ be the coordinate basis vectors, and let $\vec{u} = d/d\tau$ be the tangent vector field to the geodesics (i.e. $\vec{u} = \vec{e}_0$). But by construction at $\tau = 0$:

$$\vec{u} \cdot \vec{e}_i = \vec{e}_0 \cdot \vec{e}_i = g_{0i} = 0 \tag{88}$$

and

$$\frac{d(\vec{u} \cdot \vec{e}_i)}{d\tau} = \nabla_{\vec{u}}(\vec{u} \cdot \vec{e}_i) = 0 + \vec{u} \cdot \nabla_{\vec{e}_i}\vec{u} \tag{89}$$

(the curves are geodesics so $\nabla_{\vec{u}}\vec{u} = 0$ and $\vec{e}_i$ and $\vec{u}$ form a coordinate basis ($[\vec{e}_i, \vec{u}] = 0$. and because:

$$\vec{u} \cdot \nabla_{\vec{e}_i}\vec{u} = \frac{1}{2}\nabla_{\vec{e}_i}(\vec{u} \cdot \vec{u}) = 0 \tag{90}$$

and consequently $\vec{u} \cdot \vec{e}_i = g_{0i} = 0$ everywhere and we can write the metric in the so called synchronous form:

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{91}$$