Lesson 7
Weak fields

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Geometrized units

The units system where $c = G = 1$ is extremely useful and called geometrized units.

Quantity	Conventional Units	Geometrized Units
speed of light, c	2.998×10^8 m sec^{-1}	one
Newton's gravitation constant, G	6.673×10^{-11}m^3 kg^{-1} sec^{-2}	one
G/c^2	7.425×10^{-28} m kg^{-1}	one
c^5/G	3.629×10^{52} W	one
$c^2/\sqrt{G}$	3.479×10^{24} gauss cm $= 1.160 \times 10^{24}$ volts	one
Planck's reduced constant $\hbar$	1.055×10^{-34} kg m^{2}s^{-1}	$(1.616 \times 10^{-35}$ m$)^2$
sun's mass, $M_\odot$	1.989×10^{30} kg	1.477 km
sun's radius, $R_\odot$	6.960×10^8 m	6.960×10^8 m
earth's mass, $M_\oplus$	5.977×10^{24} kg	4.438 mm
earth's radius, $R_\oplus$	6.371×10^6 m	6.371×10^6 m
Hubble constant H_o	65 ± 25 km sec^{-1}Mpc^{-1}	$[(12 \pm 5) \times 10^9$ lt yr$]^{-1}$
density to close universe, ρ_{crit}	$9^{+11}_{-5} \times 10^{-27}$ kg m^{-3}	$7^{+8}_{-3} \times 10^{-54}$ m^{-2}

And the dimensionality can be compared like this:.

Quantity	SI dimension	Geometric dimension	Multiplication factor
Length	[L]	[L]	1
Time	[T]	[L]	c
Mass	[M]	[L]	$G\, c^{-2}$
Velocity	$[\mathrm{L\, T^{-1}}]$	1	c^{-1}
Angular velocity	$[\mathrm{T^{-1}}]$	$[\mathrm{L^{-1}}]$	c^{-1}
Acceleration	$[\mathrm{L\, T^{-2}}]$	$[\mathrm{L^{-1}}]$	c^{-2}
Energy	$[\mathrm{M\, L^2\, T^{-2}}]$	[L]	$G\, c^{-4}$
Energy density	$[\mathrm{M\, L^{-1}\, T^{-2}}]$	$[\mathrm{L^{-2}}]$	$G\, c^{-4}$
Angular momentum	$[\mathrm{M\, L^2\, T^{-1}}]$	$[\mathrm{L^2}]$	$G\, c^{-3}$
Force	$[\mathrm{M\, L\, T^{-2}}]$	1	$G\, c^{-4}$
Power	$[\mathrm{M\, L^2\, T^{-3}}]$	1	$G\, c^{-5}$
Pressure	$[\mathrm{M\, L^{-1}\, T^{-2}}]$	$[\mathrm{L^{-2}}]$	$G\, c^{-4}$
Density	$[\mathrm{M\, L^{-3}}]$	$[\mathrm{L^{-2}}]$	$G\, c^{-2}$

The weak field limit

The discussion following is relevant as to what we should expect from so-called alternative theories of gravity. The reality check we will always use is the low speed limit in which we know Newtonian physics works with admirable precision. Far away enough from the source a gravitational field should be weak in such a manner that the metric can be described:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{1}$$

where $|h_{\alpha\beta}| \ll 1$ everywhere, and $\eta_{\alpha\beta}$ is the flat Minkowski metric. What we are actually saying is that there exist coordinates in which the equation above is possible. And if this equation is true in one of these systems, then there are many other coordinate systems in which this is true. A wise choice of coordinate system is crucial.

Background Lorentz Transformations

The Lorentz transformations are:

$$\Lambda^{\bar{\alpha}}{}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = (1 - v^2)^{-\frac{1}{2}} \tag{2}$$

A Lorentz transformation is one:

$$x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} x^{\beta} \tag{3}$$

Although we are not in SR, let's see what happens to the metric:

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} g_{\mu\nu} \tag{4}$$

$$= \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} \eta_{\mu\nu} + \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} h_{\mu\nu} \tag{5}$$

But by definition of Lorentz transformations:

$$\Lambda^{\mu}{}_{\bar{\alpha}}\Lambda^{\nu}{}_{\bar{\beta}}\eta_{\mu\nu} = \eta_{\bar{\alpha}\bar{\beta}} \tag{6}$$

So:

$$g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}} + h_{\bar{\alpha}\bar{\beta}} \tag{7}$$

where:

$$h_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}}\Lambda^{\nu}{}_{\bar{\beta}}h_{\mu\nu} \tag{8}$$

which show that $h_{\mu\nu}$ transforms as if a tensor in SR itself. This property of the slightly "curved" or modified Minkowski will make it easier the calculations. All physical fields, including the Riemman tensor will be written just in terms of it.

A type of coordinates that leave equation (1) with the condition $|h_{\alpha\beta}| \ll 1$ unchanged is a small change in the coordinates of the form:

$$x^{\alpha'} = x^{\alpha} + \xi^{\alpha}(x^{\beta}), \tag{9}$$

We assume ξ^{α} is small in the sense that $|\xi^{\alpha}{}_{,\beta}| \ll 1$

$$\Lambda^{\alpha'}{}_{\beta} = \frac{\partial x'}{\partial x^{\beta}} = \delta^{\alpha}{}_{\beta} + \xi^{\alpha}{}_{,\beta}, \tag{10}$$

$$\Lambda^{\alpha}{}_{\beta'} = \delta^{\alpha}{}_{\beta} - \xi^{\alpha}{}_{,\beta} + O(|\xi^{\alpha}{}_{,\beta}|^{2}). \tag{11}$$

To first order this gives:

$$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} + \xi_{\beta,\alpha}, \tag{12}$$

where:

$$\xi_\alpha = \eta_{\alpha\beta}\xi^\beta \tag{13}$$

The effect of this change of coordinates is to change $h_{\alpha\beta}$:

$$h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}, \tag{14}$$

If $|\xi^\alpha{}_{,\beta}| \ll 1$ then the new $h_{\alpha\beta}$ is also small. A change like this is called: a **gauge transformation**. This freedom of the Einstein's theory is extremely useful and important.

The Riemann tensor:
Using (1) the calculation yields:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) \tag{15}$$

The trick or magic: i.e. the fact that the Riemann tensor does not depend on the small quantities $\xi_{\alpha,\beta}$ is precisely the fact that they are second order, and we don't need to consider them at first order.

Weak-field Einstein equations
We will define

$$h^{\mu}_{\beta} := \eta^{\mu\alpha} h_{\alpha\beta}, \tag{16}$$

$$h^{\mu\nu} := \eta^{\nu\beta} h^{\mu}_{\beta}, \tag{17}$$

the trace:

$$h := h^{\alpha}_{\alpha} \tag{18}$$

and another tensor called the "trace reverse" of $h_{\alpha\beta}$

$$\bar{h}^{\alpha\beta} := h^{\alpha_\beta} - \frac{1}{2}\eta^{\alpha\beta}h. \tag{19}$$

The trace is:

$$\bar{h} := \bar{h}^{\alpha}_{\alpha} = h^{\alpha\beta}\eta_{\beta\alpha} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\beta\alpha}h = h - \frac{4}{2}h = -h \tag{20}$$

and the inverse of (19) is the same equation:

$$h^{\alpha\beta} = \bar{h}^{\alpha_\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h}. \tag{21}$$

The Einstein tensor becomes:

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} \tag{22}$$

$$-\bar{h}_{\beta\mu,\alpha}{}^{,\mu} + O(h^2_{\alpha\beta})]. \tag{23}$$

Things would be simpler if we required:

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0. \tag{24}$$

Notice, that from the definition of $\bar{h}^{\alpha\beta}$ in eq. (19), we get:

$$h^{\alpha\beta}{}_{,\beta} - \frac{1}{2} h_{,\alpha} = 0. \tag{25}$$

We just need to choose coordinates where these equations, (24) and (25), would be satisfied.
Eq (24) is called the Lorentz gauge. Let's assume we have an $\bar{h}^{\alpha\beta}$ for which this does not hold. We look for a new one:

$$\bar{h}^{(new)}_{\mu\nu} = \bar{h}^{(old)}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \tag{26}$$

The divergence of this new $\bar{h}^{(new)}_{\mu\nu}$ will be:

$$\bar{h}^{(new)\mu\nu}{}_{,\nu} = \bar{h}^{(old)\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}{}_{,\nu}. \quad (27)$$

But then all we need is:

$$\Box\xi^{\mu} = \xi^{\mu,\nu}{}_{,\nu} = \bar{h}^{(old)\mu\nu}{}_{,\nu} \quad (28)$$

Notice that the $\Box$ operator above is nothing else than the standard Laplacian operator minus the second time derivative, i.e. for any function f:

$$\Box f = f^{,\mu}{}_{,\mu} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) f. \quad (29)$$

The inhomogeneous form of this equation always have a solution provided the function that makes the equation inhomogeneous is "well behaved". But the solution will be defined up to any other function η which satisfies:

$$\Box \eta^{\mu} = 0 \tag{30}$$

This function will of course satisfy equation (28). This procedure defines rather than a gauge, a class of gauges. In this gauge:

$$G^{\alpha\beta} = -\frac{1}{2} \Box \bar{h}^{\alpha\beta} \tag{31}$$

And consequently the weak field Einstein equations become:

$$\Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu} \tag{32}$$

These are also called the field equations in the "linearized" theory.

But notice that in vacuum these equations are:

$$G^{\alpha\beta} = \Box \bar{h}^{\alpha\beta} = 0 \tag{33}$$

But this means that if we assume a time dependence for the metric perturbation $\bar{h}^{\alpha\beta}$ it does satisfy the wave equation if its a solution of Einstein's equations!

Newtonian limit
In the Newtonian case we expect $|\phi| \ll 1$ and $|\mathrm{v}| \ll 1$. This will also mean that $|T^{00}| \ll |T^{0i}| \ll |T^{ij}|$

$$\Box \bar{h}^{00} = -16\pi\rho \tag{34}$$

We will be working in what is called the slow-motion approximation we will assume for any function f:

$$\frac{v}{c}\frac{\partial f}{\partial x^{\alpha}} \sim \frac{\partial f}{\partial x^{0}} \tag{35}$$

In this limit we use also that $T^{00} = \rho + O(\rho v^2)$ and these two conditions lead to:

$$\Box = \nabla^2 + O\left((\frac{v}{c})^2 \nabla^2 \right). \tag{36}$$

and then our equation to lowest order:

$$\nabla^2 \bar{h}^{00} = -16\pi\rho \tag{37}$$

where we can compare with Newton's equation:

$$\nabla^2 \phi = 4\pi\rho \tag{38}$$

where we take $G = 1$.

We need to identify then:

$$\bar{h}^{00} = -4\phi \tag{39}$$

We will consider all other components of $\bar{h}^{\alpha\beta} \sim 0$ so we will have:

$$h = h^{\alpha}{}_{\alpha} = -\bar{h}^{\alpha}{}_{\alpha} = \bar{h}^{00} \tag{40}$$

And then

$$h^{00} = -2\phi \tag{41}$$

$$h^{xx} = h^{yy} = h^{zz} = -2\phi \tag{42}$$

and:

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2). \tag{43}$$

Care should be taken when trying to identify (31) with the gravitational field far from the source. Notice that if we are far from the source the metric has to be a solution of the vacuum Einstein's field equations. So in these terms it would an affirmation incompatible with equation (31).

An asymptotically flat spacetime is a Lorentzian manifold in which, the curvature becomes negligible or directly vanishes at large distances from some region, so that at large distances, the geometry becomes indistinguishable from that of Minkowski spacetime. For completeness, a coordinate dependent definition of "asymptotic flatness": It's one with $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ such if $r^2 = x^2 + y^2 + z^2$ $h_{\alpha\beta}$ behaves:

$$\lim_{r\to\infty} h_{\alpha\beta} = O(1/r)$$

$$\lim_{r\to\infty} h_{\alpha\beta,\mu} = O(1/r^2)$$

$$\lim_{r\to\infty} h_{\alpha\beta,\mu\nu} = O(1/r^3)$$

But to identify (31) with a general field far from the source you have to follow the discussion in Schutz which I will not repeat here. But in conclusion, considering that far from the source the potential is:

$$\phi_{\mathit{far\ field}} = -M/r + O(r^2) \tag{44}$$

we can redefine (34) with this potential and we have:

$$ds^2 = -(1 - 2M/r)dt^2 + O(r^{-2})(dx^2 + dy^2 + dz^2). \tag{45}$$

The Weyl tensor is a tensor associated with the Riemman tensor that could be quite useful. In 4 dimensions the Riemman tensor has twenty independent components. Ten are actually given by R_{ab}; the other ten by the Weyl tensor, which is defined in n dimensions:

$$C_{abcd} = R_{abcd} + \frac{1}{n-2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{(n-1)(n-2)}(g_{ac}g_{db} - g_{ad}g_{cb})R$$

So in four dimensions which is the main interest of this course:

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R \quad (44)$$

It has the same symmetries that the Riemman tensor has:

$$
\begin{aligned}
C_{abcd} = -C_{adbc} = -C_{bacd} = C_{cdab} \\
C_{abcd} + C_{adbc} + C_{acdb} \equiv 0
\end{aligned} \tag{45}
$$

but it also has an additional symmetry:

$$
{C^a}_{bad} \equiv 0 \tag{46}
$$

The Weyl tensor is trace free.

Conformal metrics

Two metrics are said to be conformally related if:

$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{47}$$

where Ω is non zero differentiable function of the coordinates. In such geometries angles between vectors and ratio of magnitudes between vectors are the same for each metric. Also the null geodesics coincide. Additionally they have the same Weyl tensor:

$$\bar{C}^a_{bcd} = C^a{}_{bcd} \tag{48}$$

Any quantity which satisfies relationships like (47) are called *conformally invariant.* A metric is said to be conformally flat if it can be reduced to:

$$g_{ab} = \Omega^2 \eta_{ab} \tag{49}$$

where η_{ab} is the flat metric.

The following are two important Theorems for which we will state the results only:

Theorem I

A necessary and sufficient condition for a metric to be conformally flat is that its Weyl tensor vanishes everywhere.

Theorem II

Any two dimensional Riemmanian manifold is conformally flat.

The Newtonian limit of non-Einstein theories

A metric theory (devised by Nordstrom in 1913) relates $g_{\mu\nu}$ and $T_{\mu\nu}$ by:

$$C_{\mu\nu\rho\sigma} = 0$$
$$R = \kappa g_{\mu\nu} T^{\mu\nu} \tag{50}$$

where C is the Weyl tensor. We will show that this theory, in the proper Newtonian limit and with the proper choice of κ, agrees with Newtonian gravitation theory, but predicts no deflection of starlight passing near the Sun. We can, due to the vanishing of the Weyl tensor write the metric:

$$g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu} \tag{51}$$

where $\phi \ll 1$ in the Newtonian limit.

It can be proven that for general metrics like (51) the Ricci scalar is:

$$R = R^{\mu}_{\mu} = e^{2\phi}\eta^{\mu\nu}R_{\mu\nu} = -6e^{-2\phi}[\nabla^2\phi + (\nabla\phi)^2] \qquad (52)$$

Then:

$$R \approx -6\nabla^2\phi \qquad (53)$$

For low speeds ${T^{\mu}}_{\mu} \approx {T^0}_0 \approx -\rho$ and the field equations become:

$$-6\nabla^2\phi_{,\alpha\beta}\eta^{\alpha\beta} = \kappa T = -\kappa\rho \qquad (54)$$

Time variations will be slow compared to space variations at low speeds:

$$6\nabla^2\phi_{,ij}\delta^{ij} = \kappa T = \kappa\rho \tag{55}$$

For $\kappa = 24\pi$ these are the Newton equations.
For the Newtonian limit (55) gives

$$g_{00} = -(1 + 2\phi)$$

This means that the trajectories are geodesics of the metric.
We could see that the null geodesics for metric (51) are the same null geodesics of the Minkowski metric.
This means that far from the source we will see no deflection near the a massive object.

Regarding another predicted effect of Einstein's theory, the Pound-Rebka experiment of the gravitational redshift:
Near the Earth's surface the metric is of the form,

$$ds^2 = e^{2\phi(z)}(-dt^2 + dx^2 + dy^2 + dz^2) \tag{56}$$

For particles to fall the known acceleration we need to have $\phi \approx -gz$ From the geodesic equation we find that for the energy of a photon moving vertically

$$\frac{dp^0}{dz} = -\Gamma^0{}_{0z} p^0 = -\phi_{,z} p^0 \tag{57}$$

So the photon will loose energy and the theory predicts a gravitational redshift.